UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS AND STATISTICS
Math122–001 Linear Algebra I
Solutions to Assignment 2: CORRECTED

1. In each part, we have to find the equation of a line, so we need both a point on the line, and its direction vector.

   (a) The plane $2x-2y+z-5=0$ has normal vector $\mathbf{n}=(2,-2,1)$ (i.e. the coefficients of x, y and z). Thus $\mathbf{n}$ is the direction vector of the line we need. Since this line passes through $P(1,6,-1)$, the parametric of this line is

   $$(1,6,-1)+t(2,-2,1) \quad \text{where } t\in\mathbb{R},$$

   that is,

   $$\left.\begin{array}{rcl} x &=& 1+2t \\ y &=& 6-2t \\ z &=& -1+t \end{array}\right\} \text{where } t\in\mathbb{R}.$$

   (5 marks)

   (b) Any line orthogonal to ℓ_1 and ℓ_2 must be orthogonal to their direction vectors. So we can obtain the direction vector of our new line by taking a cross product. Let $\mathbf{u}=(2,-1,3)$ and $\mathbf{v}=(-2,3,2)$ denote the direction vectors of ℓ_1 and ℓ_2 respectively. Then

   $$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -2 & 3 & 2 \end{vmatrix} \\ &= (-2-9)\mathbf{i}-(4+6)\mathbf{j}+(6-2)\mathbf{k} \\ &= -11\mathbf{i}-10\mathbf{j}+4\mathbf{k} \\ &= (-11,-10,4). \end{aligned}$$

   Also, our line passes through $P(1,2,3)$, so its parametric equation must be

   $$(1,2,3)+t(-11,-10,4) \quad \text{where } t\in\mathbb{R},$$

   that is,

   $$\left.\begin{array}{rcl} x &=& 1-11t \\ y &=& 2-10t \\ z &=& 3+4t \end{array}\right\} \text{where } t\in\mathbb{R}.$$

   (12 marks)

2. (a) If a plane is orthogonal to a line, the normal vector of that plane will be the direction vector of that line. For the line we are given, the direction vector is $(-1,2,5)$, so this is the normal vector of our plane.

   Therefore the equation of our plane is

   $$-x+2y+5z+d=0,$$

and to find d we substitute the values of a point we know to be on this plane. As we are told that it passes through the point $P(1,0,4)$, we substitute $x=1$, $y=0$ and $z=4$, which gives $d=-19$. So the equation we need is

$$-x+2y+5z-19=0.$$

(6 marks)

(b) To determine if a point lies on a plane, we substitute its coordinates into the plane equation, and see it it's satisfied. Now, for the plane in part (a) and the point $Q(2,2,-2)$, we have

$$-2+(2\times 2)+(5\times -2)-19=-2+4-10-19=-27\neq 0,$$

so Q does not lie on this plane. But for $R(0,2,3)$, we have

$$-0+(2\times 2)+(5\times 3)-19=4+15-19=0,$$

so R *does* lie on this plane.

(4 marks)

3. (a) To find the intersection of a line and a plane, we substitute the parametric equation of the line into the equation of the plane, and solve for t.

   i. First, we have

$$\begin{aligned} & (1+t)-2(1-2t)+3(3-2t)-2=0 \\ \iff\ & (1-2+9-2)+t(1+4-6)=0 \\ \iff\ & 6-t=0 \\ \iff\ & t=6. \end{aligned}$$

   Thus the line ℓ_1 intersects the plane at the point P where $t=6$, i.e. the point $P(7,-11,-9)$.

   ii. Second, we have

$$\begin{aligned} & (-1+t)-2(-1-t)+3(-1-t)-2=0 \\ \iff\ & (-1+2-3-2)+t(1+2-3)=0 \\ \iff\ & -4=0, \end{aligned}$$

   which is impossible. Hence there is no value of t for which the point on the line ℓ_2 satisfies the plane equation, and thus ℓ_2 does not intersect our plane. (This, of course, means that they must be parallel.)

   iii. Finally, we have

$$\begin{aligned} & (1+t)-2(1-t)+3(1-t)-2=0 \\ \iff\ & (1-2+3-2)+t(1+2-3)=0 \\ \iff\ & 0=0, \end{aligned}$$

   which holds regardless of the value of t. Therefore any point on the line ℓ_3 must also be on the plane, i.e. the line ℓ_3 is contained within the plane.

(12 marks)

(b) If a line and a plane in $\mathbb{R}^3$ are orthogonal, they must intersect in a unique point. Thus we only need consider case (i). Also, the line must be in the same direction as the normal vector of the plane. However, our plane has normal vector $(1,-2,3)$, while the direction vector of ℓ_1 is $(1,-2,-2)$; clearly these have different directions (as $(1,-2,3)$ is not a scalar multiple of $(1,-2,-2)$).

So none of the lines in part (a) are orthogonal to the plane $x-2y+3z-2=0$.

(5 marks)

4. (a) To find the equation of the plane in $\mathbb{R}^3$ through the given points, we first find its normal vector $\mathbf{n}$. We can do this by taking the cross product of $\overrightarrow{PQ}=(0,3,-1)$ and $\overrightarrow{PR}=(-2,1,2)$. Now

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3 & -1\\ -2 & 1 & 2\end{vmatrix}\\ &= (6+1)\mathbf{i}-(0-2)\mathbf{j}+(0+6)\mathbf{k}\\ &= 7\mathbf{i}+2\mathbf{j}+6\mathbf{k}\\ &= (7,2,6).\end{aligned}$$

Thus the equation of the plane is

$$7x+2y+6z+d=0.$$

To find d, we take any point on the plane and substitute that into our equation and solve for d. We might as well use $Q=Q(1,1,0)$ (although P or R would also work). So we have

$$(7\times 1)+(2\times 1)+(6\times 0)+d=0,$$

and rearranging this gives $d=-9$, so the equation of the plane is

$$7x+2y+6z-9=0.$$

(9 marks)

(b) The problem with the points $P(0,3,3)$, $Q(-2,-1,-3)$ and $R(1,5,6)$ is that they all lie on the same line (that is, they are *collinear*). This can be seen from the fact that the vectors $\overrightarrow{PR}=(1,2,3)$ and $\overrightarrow{PQ}=(-2,-4,-6)$ are scalar multiples, and therefore have the same direction. Consequently, any plane containing this line will contain the three points, and so there can't possibly be a *unique* plane passing through them.

Alternatively, we notice that the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$ is the zero vector, which can't possibly occur as the normal vector to a plane, so the method collapses here, too.

(7 marks)

5. (a) For each system of linear equations, we'll use Gauss–Jordan elimination to put the augmented matrix of the system into reduced row-echelon form (RREF), and obtain the general solution from that.

i. For the system

$$\begin{aligned} x-2y+\ z &= 3 \\ 2x+2y-4z &= 0 \end{aligned}$$

we have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & 4 & -4 & 0 \end{array}\right].$$

Using row operations, we obtain

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & 4 & -4 & 0 \end{array}\right] \underset{R_2-2R_1\to R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & -6 & -6 \end{array}\right] \underset{\frac{1}{6}R_2\to R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$$\underset{R_1+2R_2\to R_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right],$$

which is in RREF.
Now, the column corresponding to the variable z does not contain a leading 1, so is a free variable, and we set $z=t$. From this we obtain

$$\begin{aligned} x &= 1+t \\ y &= -1+t \\ z &= t \end{aligned}$$

so our general solution is $(x,y,z)=(1,-1,0)+t(1,1,1)$ (where $t\in\mathbb{R}$).

(8 marks)

ii. For the system

$$\begin{aligned} x+2y-\ z &= 3 \\ -2x-4y+2z &= 11 \end{aligned}$$

we have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ -2 & -4 & 2 & 11 \end{array}\right].$$

Using row operations, we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ -2 & -4 & 2 & 11 \end{array}\right] \underset{R_2+2R_1\to R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 17 \end{array}\right],$$

and although this isn't in RREF yet, we can already see that there is *no solution* to our linear system.

(5 marks)

iii. For the system

$$\begin{aligned} x-2y+\ z &= 2 \\ 3x+4y-7z &= 4 \\ 2x+\ y-3z &= 3 \end{aligned}$$

we have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & \mathbf{2} \\ 3 & 4 & -7 & 4 \\ 2 & 1 & -3 & 3 \end{array}\right].$$

So using elementary row operations, we have:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & \mathbf{2} \\ 3 & 4 & -7 & 4 \\ 2 & 1 & -3 & 3 \end{array}\right] \xrightarrow[\substack{R_2-3R_1\to R_2 \\ R_3-2R_1\to R_3}]{} \left[\begin{array}{ccc|c} 1 & -2 & 1 & \mathbf{2} \\ 0 & 10 & -10 & -2 \\ 0 & 5 & -5 & -1 \end{array}\right] \xrightarrow[\frac{1}{10}R_2\to R_2]{} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{5} \\ 0 & 5 & -5 & -1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1+2R_2\to R_1 \\ R_3-5R_2\to R_3}]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{8}{5} \\ 0 & 1 & -1 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which is in RREF. Now, the column corresponding to the variable z does not contain a leading 1, so is therefore a free variable, and we set $z = t$. From this we obtain

$$\begin{aligned} x &= \frac{8}{5} + t \\ y &= -\frac{1}{5} + t \\ z &= t \end{aligned}$$

so our general solution is $(x,y,z) = (\frac{8}{5}, -\frac{1}{5}, 0) + t(1,1,1)$ (where $t \in \mathbb{R}$).

(9 marks)

iv. Finally, for the system of linear equations

$$\begin{aligned} -2x - 4y - z &= 1 \\ 2x + \ y + z &= 2 \\ x + 2y + z &= 0 \end{aligned}$$

we have the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -4 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{array}\right].$$

(Note that sometimes, we can deviate from the exact sequence of elemenrary row operations specified by Gauss–Jordan elimination, if it helps make our calculations easier. I'm going to start by interchanging rows 1 and 3, rather than dividing row 1 by -2, in order to avoid having too many fractions.)

$$\left[\begin{array}{ccc|c} -2 & -4 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_1\leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ -2 & -4 & -1 & 1 \end{array}\right] \xrightarrow[\substack{R_2-2R_1\to R_2 \\ R_3+2R_1\to R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_2\to R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1-2R_2\to R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1-\frac{1}{3}R_3\to R_1\\ R_2-\frac{1}{3}R_3\to R_2}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

which is in RREF. (Of course, if you did a different sequence of row operations, you should still have the same RREF, since the RREF of a matrix is unique.)
From this we can see that our system of linear equations has a unique solution:

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 1. \end{aligned}$$

(8 marks)

(b) Of the four systems of linear equations in part (a), the systems in (i), (iii) and (iv) are consistent, as we have either a unique solution (as in (iv)) or infinitely many solutions (in (i) and (iii)). Only the system in (ii) is inconsistent, where we have no solution.

(4 marks)

(c) In (i), we have two planes intersecting along a line. The equation of that line is

$$\begin{aligned} x &= 1+t \\ y &= -1+t \\ z &= t \end{aligned}$$

In (iii), we have three planes intersecting along a common line, and that line has equation

$$\begin{aligned} x &= \frac{8}{5}+t \\ y &= -\frac{1}{5}+t \\ z &= t. \end{aligned}$$

(Note that we really do have three *distinct* planes: looking at their equations, none is a scalar multiple of another.)
In (iv), we have three planes intersecting at a unique point, and that point must be $(1,-1,1)$.

(6 marks)

Total marks: 100

6. I'll use the notation $\langle\langle A,B,C\rangle\rangle$ to denote the plane in $\mathbb{R}^3$ containing the three points A,B,C.

First of all, as the room has a completely flat floor, we can think of that as a plane, Π.

Now, suppose the ends of the three legs of the stool are points A, B and C. Now, assuming those points are not collinear (which is unlikely in a piece of furniture), they will specify a unique plane. By placing the stool on the floor, the plane $\langle\langle A,B,C\rangle\rangle$ will coincide with the plane Π. Consequently, the stool will be stable.

On the other hand, the chair has four legs. Let the ends of the legs be the points P, Q, R and S (listed going clockwise, when looking down from above), and suppose the legs are positioned at the corners of the chair. Now, P,Q,R,S need not all lie on the same plane (i.e. they need not be *coplanar*): for instance, S may not lie on the plane $\langle\langle P,Q,R\rangle\rangle$. In a nice, newly-made chair, one would expect it to be designed so that they were coplanar, but in an old, worn-out chair we can no longer assume this. For instance, it could happen so that when the legs ending at P, Q and R are placed on the floor (so that Π corresponds with $\langle\langle P,Q,R\rangle\rangle$), the point S is in mid-air somewhere. If we pushed down hard enough on that corner, then the legs ending at P, R and S would meet the floor instead, and so the plane $\langle\langle P,R,S\rangle\rangle$ would coincide with Π. The chair can pivot about an axis through the line PR, and is not stable.

(Up to 20 bonus marks for a complete solution)

R. F. Bailey, 2 February 2012; revised 8 February 2012