Some Notes on Distance-Transitive and Distance-Regular Graphs

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These are notes from lectures given in the Queen Mary Combinatorics Study Group on 13th and 20th February 2004, and also 5th March 2004. They are based on the author's M.Math. project, which can be found at:

`http://www.maths.qmul.ac.uk/~rfb/dtg.pdf`.

1 Introductory Definitions

In these notes, $\Gamma = (V\Gamma, E\Gamma)$ denotes a graph, as we will use G to denote a group. All graphs considered will be simple, finite, connected and undirected.

Definition
An *automorphism* of Γ is a bijective function $g : V\Gamma \to V\Gamma$ such that $v \sim w$ if and only if $g(v) \sim g(w)$. The set of all automorphisms is the *automorphism group* of Γ, denoted by $\mathrm{Aut}(\Gamma)$. If, for all $u, v \in V\Gamma$, there exists some $g \in \mathrm{Aut}(\Gamma)$ such that $g(u) = v$, the Γ is *vertex-transitive*.

Examples of vertex-transitive graphs include the *n*-circuits (not very exciting), and (slightly more exciting) the *Petersen graph*, pictured below.

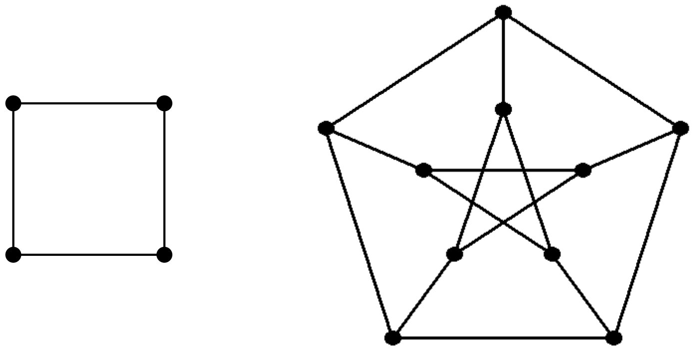


Figure 1: The 4-circuit (left) and Petersen graph (right)

A non-example is the complete bipartite graph $K_{1,n}$, as no automorphism can map the central vertex of degree n to an outer vertex of degree 1. $K_{1,4}$ is pictured below.

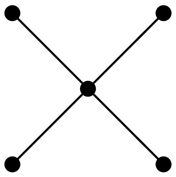


Figure 2: The complete bipartite graph $K_{1,4}$

Definition
For vertices $u,v \in V\Gamma$, the *distance* from u to v is defined to be the least number of edges in a path from u to v, and is denoted by $d(u,v)$.

We have:

- $d(u,v) \geq 0$, with $d(u,v) = 0 \Leftrightarrow u = v$;
- $d(u,v) = d(v,u)$ (as Γ is undirected);
- $d(u,w) \leq d(u,v) + d(v,w)$.

Hence $(\Gamma, d(\cdot,\cdot))$ is a metric space.

The maximum distance in Γ is the *diameter* of Γ, denoted diam(Γ).

Definition
Γ is *distance-transitive* if, for vertices $u,v,w,x \in V\Gamma$ with $d(u,v) = d(w,x)$, there exists some $g \in \text{Aut}(\Gamma)$ satisfying $g(u) = w$ and $g(v) = x$.

(This does not mean that Aut(Γ) is 2-transitive, unless Γ is a complete graph.)

Clearly, we have that distance-transitivity implies vertex-transitivity (consider two pairs of vertices at distance 0).

The two vertex-transitive graphs in figure 1 are both distance-transitive. However, the following graph is vertex-transitive but *not* distance-transitive. The cyclic n-ladder L_n (L_6 is shown below) is clearly vertex-transitive – we can rotate and reflect L_n; we can also move any of the 'inner ring' to anywhere on the 'outer ring'. However, Γ is not distance-transitive: consider u,v,u',v' as shown. Clearly $d(u,v) = d(u',v') = 2$. But there is no automorphism that moves $\{u,v\}$ to $\{u',v'\}$, as there is only one geodesic (shortest) path from u to v, while there are two from u' to v'.

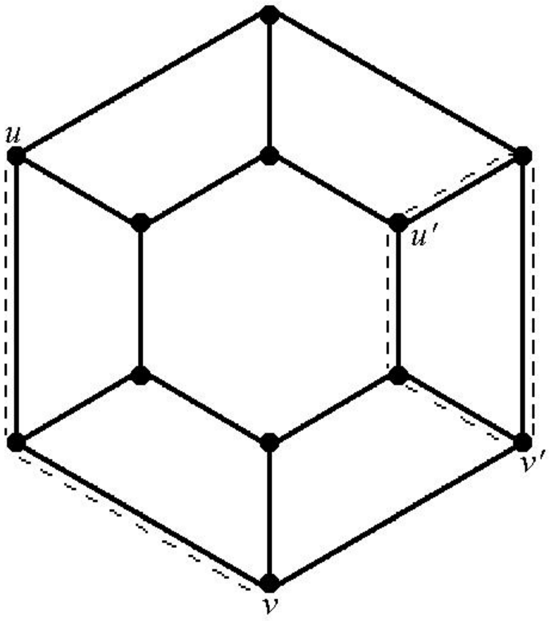


Figure 3: A vertex-transitive graph that is not distance-transitive

2 Intersection Arrays

These are the parameters which are seen most often when distance-transitive (or distance-regular) graphs are being considered.

Definition
For a graph Γ, vertex $v \in V\Gamma$, and for $0 \leq i \leq d = \mathrm{diam}(\Gamma)$, define

$$\Gamma_i(v) = \{w \in V \ : \ d(v,w) = i\}.$$

These *cells* $\Gamma_0(v), \Gamma_1(v), \ldots, \Gamma_d(v)$ form a *distance partition* (or a *level decomposition*) of Γ. For example:

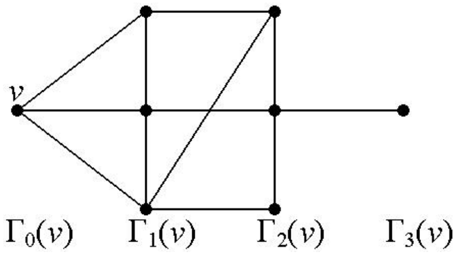


Figure 4: A distance partition

We can use distance partitions to provide an alternative characterisation of distance-transitivity.

Proposition
Suppose Γ is connected, has $\mathrm{diam}(\Gamma) = d$ and automorphism group $\mathrm{Aut}(\Gamma) = G$. Then Γ is distance-transitive if and only if it is vertex-transitive and $\mathrm{Stab}_G(v)$ is transitive on $\Gamma_i(v)$ for $i = 1, \ldots, d$ and for all $v \in V\Gamma$.

Proof:
First, suppose Γ is distance-transitive, so Γ is also vertex-transitive. Consider $u, u' \in \Gamma_i(v)$, i.e. with $d(u,v) = d(u',v) = i$. Then there exists an automorphism $g \in G$ with $vg = v$ and $ug = u'$. Thus $g \in \mathrm{Stab}_G(v)$, and $\mathrm{Stab}_G(v)$ is transitive on $\Gamma_i(v)$.

Conversely, suppose Γ is vertex-transitive and that $\mathrm{Stab}_G(v)$ is transitive on $\Gamma_i(v)$. Consider $u, w, u', w' \in V\Gamma$, such that $d(u,w) = d(u',w') = i$. Let $g \in G$ be such that $wg = w'$ and choose $h \in \mathrm{Stab}_G(w')$ so that $(ug)h = u'$. Then for the composition gh, we get $u(gh) = u'$ and $w(gh) = (w')h = w'$. So gh is an automorphism taking $\{u, w\}$ to $\{u', w'\}$. Hence Γ is distance-transitive. □

Now, for any graph Γ and $u, v, \in V\Gamma$, we can define *intersection numbers*

$$\begin{aligned} s_{hi}(u,v) &= |\{w \in V\Gamma \,:\, d(u,w) = h, d(v,w) = i\} \\ &= |\Gamma_h(u) \cap \Gamma_i(v)|. \end{aligned}$$

In a distance-transitive graph, these numbers depend only on $d(u,v)$ (and not on the individual vertices), so are constants. Hence, if $d(u,v) = j$, we write s_{hij} for $s_{hi}(u,v)$. These show that a distance-transitive graph is an *association scheme* on $V\Gamma$, where u and v are i^{th} *associates* if and only if $d(u,v) = i$.

We are particularly interested in the intersection numbers when $h = 1$. If $d(u,v) = j$, we have

$$s_{1ij} = |\Gamma_1(u) \cap \Gamma_i(v)|,$$

that is, the number of neighbours of u at distance i from v. Clearly the only possibilities for s_{1ij} to be non-zero are $i = j-1$, j or $j+1$. So we give these values of s_{1ij} special names:

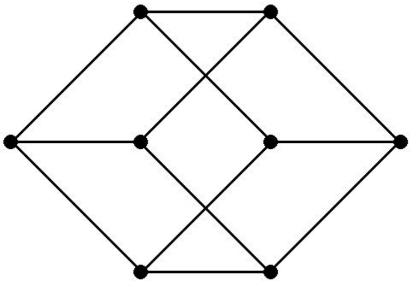
$$\begin{aligned} s_{1(j-1)j} &= c_j \\ s_{1jj} &= a_j \\ s_{1(j+1)j} &= b_j. \end{aligned}$$

We put these in an array, called the *intersection array* of Γ, denoted by $\iota(\Gamma)$:

$$\iota(\Gamma) = \left\{ \begin{array}{ccccc} * & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & * \end{array} \right\}.$$

Note that c_0 and b_d are undefined (which is what the $*$ represents). The easiest way to understand these intersection numbers is to think of a distance partition of Γ. If we fix v and choose $u \in \Gamma_j(v)$, then c_j denotes the the number of neighbours of u that are closer to v, a_j the number of neighbours at the same distance, and b_j the number that are further away.

Example



$$\iota(Q_3) = \left\{ \begin{array}{cccc} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{array} \right\}$$

Figure 5: The intersection array of the cube Q_3

Some simple observations are that $a_0 = 0$ (as there is only one vertex in $\Gamma_0(v)$, $c_1 = 1$ (for the same reason), $b_0 = k$, where k is the valency of the graph, and that all columns sum to k. A less trivial observation (which requires proof) is that the top row is weakly increasing (i.e. $1 = c_1 \leq c_2 \leq \cdots \leq c_d$) and that the bottom row is weakly descreasing (i.e. $k \geq b_1 \geq \cdots \geq b_{d-1}$). There are also a number of other numerical constraints which can be derived.

So, beginning with the assumption that Γ was distance-transitive, we have obtained some purely combinatorial parameters. But, without prior reference to its automorphism group or any transitivity properties, we could pose the question, “What if a graph happens to satisfy these parameters?”. Then we have the following:

Definition
If a graph Γ has an intersection array, we say Γ is *distance-regular*. If a distance-regular graph has diameter 2, then it is a *strongly-regular* graph.

3 Families of Distance-Transitive Graphs

3.1 Hamming graphs

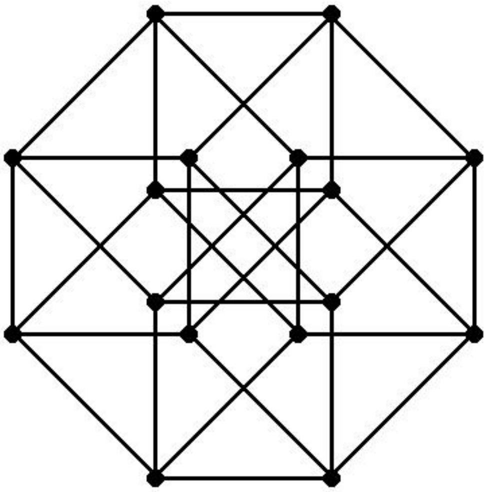
Let $X = \{1, \ldots, n\}$ be a set of size n. The *Hamming graph* $H(d,n)$ is defined as follows. The vertex set is X^d (the cartesian product of d copies of X), so the vertices are ordered d-tuple of elements of X. Two vertices u and v are adjacent if and only if (when regarded as d-tuples) they differ in one co-ordinate (i.e. their *Hamming* *distance* is 1).

The automorphism group is

$$(S_n \times S_n \times \cdots \times S_n) \rtimes S_d = S_n \,\mathrm{Wr}\, Sd$$

which not only acts transitively in each co-ordinate, but also permutes the co-ordinates transitively. Distance between vertices is exactly the Hamming distance between the corresponding d-tuples, and this is preserved by $S_n \,\mathrm{Wr}\, S_d$. It follows that $H(d,n)$ is distance-transitive. *(NB. Don't look in the project for a proof: the one given in there is incorrect, as I discovered after giving this talk!)*

Example
The Hamming graphs $H(d,2)$ are the d-dimensional *hypercubes,* Q_d. Q_4 is shown below.



$$\iota(Q_4) = \left\{ \begin{array}{ccccc} * & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & * \end{array} \right\}$$

Figure 6: The 4-cube Q_4

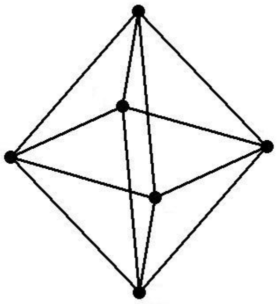
Their automorphism group $S_2 \,\mathrm{Wr}\, S_d$ is known as the *hyperoctahedral group*.

3.2 Johnson graphs

Again, we consider $X = \{1,\ldots,n\}$. Let $\mathcal{X}_k$ denote the family of all k-subsets of X; clearly $|\mathcal{X}_k| = \binom{n}{k}$. The *Johnson graph* $J(n,k)$ is defined as follows. It has vertex set $\mathcal{X}_k$; two vertices labelled by $u = \{i_1,\ldots,i_k\}$ and $v = \{j_1,\ldots,j_k\}$ are adjacent if and only if they differ in one element, i.e. if $|u \cap v| = k-1$.

The symmetric group S_n acts transitively on $\mathcal{X}_k$. It follows that $d(u,v) = m$ if and only if $|u \cap v| = k-m$, and that if $|u \cap v| = k-m$, then for $g \in S_n$, $|g(u) \cap g(v)| = k-m$. That $J(n,k)$ is distance-transitive follows from this.

Example
The Johnson graph $J(4,2)$ is the octahedron.



$$\iota(J(4,2)) = \left\{ \begin{array}{ccc} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{array} \right\}$$

Figure 7: The octahedron $J(4,2)$

3.3 Grassmann graphs

These are the so-called “q-analogues” of the Johnson graphs. Let $V = \mathbb{F}_q^n$, the n-dimensional vector space over the finite field of q elements, where q is a prime-power. The *Grassmann graph* $G(q,n,k)$ is defined as follows. The vertices are the k-dimensional subspaces of V, and two vertices are adjacent if, as subspaces, they intersect in a space of dimension $k-1$. Similar to the Johnson graphs, we have that $d(u,v) = m$ if and only if $\dim(u \cap v) = k - m$.

The projective general linear group $PGL(n,q)$ acts transitively on the vertices, and preserves the dimension of the intersection of subspaces. It follows from this that $G(q,n,k)$ is distance-transitive. (The proof is messy and not very exciting.)

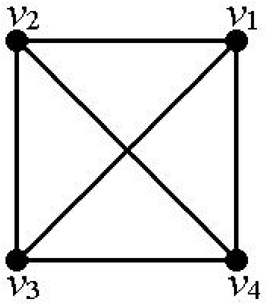
4 Algebraic Constraints on Intersection Arrays

The moral of this story is the following. Suppose a man comes up to you in a dark alley, trying to sell you what he claims is an intersection array. How can you tell if he’s telling you the truth? We use linear algebra to obtain a method which can tell us that a large number of “intersection arrays” do not correspond to any distance-regular graph.

Definition The *adjacency matrix* of a graph Γ with n vertices is the n by n matrix $\mathbf{A}(\Gamma)$ with entries

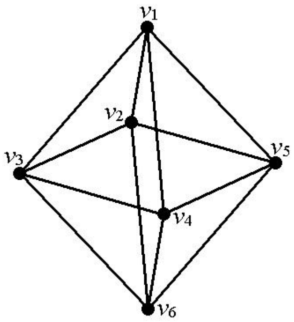
$$\mathbf{A}_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

For example, the adjacency matrices of K_4 and the octahedron are shown in figures 8 and 9.



$$\mathbf{A}(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 8: The complete graph K_4



$$\mathbf{A}(J(4,2)) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 9: The octahedron $J(4,2)$

The *eigenvalues* of Γ are defined to be those of $\mathbf{A}(\Gamma)$. The *adjacency algebra*, denoted $\mathcal{A}(\Gamma)$, of Γ is the set of all linear combinations of powers of $\mathbf{A}(\Gamma)$ (with coefficients in $\mathbb{C}$). This is a vector space where the elements (which are n by n matrices) can be multiplied together, so forms an algebra. Because each element of $\mathcal{A}(\Gamma)$ has the form $\Sigma\lambda_i\mathbf{A}^i$, this algebra is clearly commutative. In the case of a distance-regular graph, which forms an association scheme, then $\mathcal{A}(\Gamma)$ is precisely the *Bose-Mesner algebra* of that association scheme.

The powers of $\mathbf{A}$ have a graph-theoretical interpretation, as the following theorem shows.

Theorem
The number of paths in Γ from vertex v_i to vertex v_j of length l is given by the i,j^{th} entry in $\mathbf{A}^l$.

Proof:
This is done by induction on l. $\square$

If Γ is connected, this condition tells us a property of the adjacency algebra.

Corollary
If Γ is connected, and has diameter d, then the dimension of $\mathcal{A}(\Gamma)$ is at least $d+1$.

Proof:
By considering vertices along a path between to vertices at distance d, it follows that

$$\{\mathbf{A}^0, \mathbf{A}^1, \mathbf{A}^2, \ldots, \mathbf{A}^d\}$$

is a linearly independent set of size $d+1$ in $\mathcal{A}(\Gamma)$. □

We will now go on to show that in a distance-regular graph this lower bound is actually achieved.

Definition
In a connected graph Γ of diameter d, for $0 \leq i \leq d$, the i^{th} *distance matrix* $\mathbf{A}_i$ is given by

$$(\mathbf{A}_i)_{rs} = \begin{cases} 1 & \text{if } d(v_r, v_s) = i \\ 0 & \text{otherwise} \end{cases}$$

for all $v_r, v_s \in V\Gamma$.

These are a generalisation of the adjacency matrix, as clearly $\mathbf{A}_1 = \mathbf{A}(\Gamma)$. Note that $\mathbf{A}_0 = I$ (as $d(v_r, v_s) = 0$ if and only if $v_r = v_s$), and that the sum of all the distance matrices is the all-ones matrix $\mathbf{J}$. From now on, we shall only consider distance-regular graphs.

Lemma 1
For a distance-regular graph Γ, we have

$$\mathbf{A}.\mathbf{A}_i = b_{i-1}\mathbf{A}_{i-1} + a_i\mathbf{A}_i + c_{i+1}\mathbf{A}_{i+1}$$

(for $1 \leq i \leq d-1$), $\mathbf{A}.\mathbf{A}_0 = a_0\mathbf{A}_0 + c_1\mathbf{A}_1$ and $\mathbf{A}.\mathbf{A}_d = b_{d-1}\mathbf{A}_{d-1} + a_d\mathbf{A}_d$.

Proof:
Consider the r,s^{th} entry of $\mathbf{A}.\mathbf{A}_i$. We have

$$(\mathbf{A}.\mathbf{A}_i)_{rs} = |\Gamma_1(v_r) + \Gamma_i(v_s)|$$

i.e. the number of vertices adjacent to v_r and at distance i from v_s. The result follows from this. □

Lemma 2
In a distance-regular graph Γ, $\mathbf{A}_i$ can be expressed as a polynomial in $\mathbf{A}$ of degree i (and consequently belongs to $\mathcal{A}(\Gamma)$.

Proof:
This is by induction on i, applying lemma 1 in the induction step. □

Theorem
For a distance-regular graph Γ of valency k and diameter d, $\mathcal{A}(\Gamma)$ has basis $\{\mathbf{A}_0, \ldots, \mathbf{A}_d\}$ and its dimension is $d+1$.

Proof:
Clearly $\{\mathbf{A}_0, \ldots, \mathbf{A}_d\}$ is a linearly independent set, as in a given position there is a non-zero entry in only on $\mathbf{A}_i$. Now, the minimum polynomial of $\mathbf{A}$ is

$$\min(\mathbf{A}) = (\mathbf{A} - k\mathbf{I})\,\mathbf{J} = (\mathbf{A} - k\mathbf{I})\,(\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_d)\,,$$

a polynomial in $\mathbf{A}$ of degree $d+1$. Hence the dimension of $\mathcal{A}(\Gamma)$ is $d+1$. $\square$

Corollary
$\{\mathbf{A}^0, \mathbf{A}^1, \ldots, \mathbf{A}^d\}$ is another basis for $\mathcal{A}(\Gamma)$. $\square$

Now we consider the linear transformation of $\mathcal{A}(\Gamma)$ given by multiplying an element $\mathbf{X} = \sum_{i=0}^{d} s_i \mathbf{A}_i$ by $\mathbf{A}$. By lemma 1, we have

$$\mathbf{XA} = \sum_{i=0}^{d} s_i\,(b_{i-1}\mathbf{A}_{i-1} + a_i\mathbf{A}_i + c_{i+1}\mathbf{A}_{i+1})\,.$$

In other words, we can represent this linear transformation by $\mathbf{B}(\Gamma)$, the *intersection matrix*:

$$\mathbf{B}(\Gamma) = \begin{bmatrix} 0 & 1 & & & & & \\ k & a_1 & c_2 & & & & \\ & b_1 & a_2 & \cdot & & & \\ & & b_2 & \cdot & \cdot & & \\ & & & \cdot & \cdot & c_{d-1} & \\ & & & & \cdot & a_{d-1} & c_d \\ & & & & & b_{d-1} & a_d \end{bmatrix}.$$

We notice that this looks very much like the intersection array tilted slightly and with lots of zeroes thrown in the empty spaces.

Now, as this matrix represents the same linear transformation as $\mathbf{A}$ (with respect to different bases), they must have *the same eigenvalues*. Now, there are $d+1$ distinct eigenvalues, which we call $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. As eigenvalues of $\mathbf{B}$, they each have multiplicity 1. As eigenvalues of $\mathbf{A}$, they will (in general) have multiplicities greater than 1. However, it transpires quite surprisingly that these numbers can be obtained *directly from* $\mathbf{B}$.

Let $\mathbf{u}_i$ denote the *left* eigenvector of $\mathbf{B}$ corresponding to λ_i, and $\mathbf{v}_i$ the corresponding *right* eigenvector.

Theorem
The multiplicity m_i of λ_i as an eigenvector of **A** is given by

$$m_i = \frac{\langle \mathbf{u}_0, \mathbf{v}_0 \rangle}{\langle \mathbf{u}_i, \mathbf{v}_i \rangle}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product.

Proof:
Fiddly and technical. □

An important observation is that these numbers m_i must be integers. So, if we are given an array of integers which looks like it might be an intersection array of some distance-regular graph, we can transform it to the corresponding "intersection matrix", calculate its left eigenvectors and these numbers m_i. But if it transpires that one of these numbers is *not* an integer, we know immediately that there can be no distance-regular graph corresponding to this array.

Example
There is no distance-regular graph with intersection array

$$\left\{ \begin{array}{cccc} * & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & * \end{array} \right\},$$

as the set of "multiplicities" includes

$$\frac{12}{\frac{1}{3}(13 - \sqrt{13})}.$$

5 Primitive and Imprimitive Graphs

We begin this section with a digression into the theory of permutation groups. Let G be a group acting *transitively* on a set X.

Definition
A *G-congruence* is an equivalence relation $\equiv$ on X which is preserved under the action of G, that is

$$x \equiv y \Leftrightarrow g(x) \equiv g(y)$$

for all $g \in G$.

For any action, we always have the following two G-congruences:

- the *trivial* G-congruence, where $x \equiv y \Leftrightarrow x = y$;
- the *universal* G-congruence, where $x \equiv y$ for all $x, y \in X$.

Definition If there are no other G-congruences, we say the action of G on X is *primitive*. Otherwise, it is called *imprimitive*. The equivalence classes are known as *blocks*. These form a partition of X which is unaffected by the action of G.

Definition
A distance-transitive graph Γ is (im)primitive if and only if $\mathrm{Aut}(\Gamma)$ is (im)primitive on $V\Gamma$.

Examples
(i) The complete graphs K_n are primitive, as there is no partition of the vertices which is invariant under S_n.
(ii) The complete bipartite graphs $K_{n,n}$ are imprimitive; the blocks of imprimitivity are precisely the two halves of the bipartition. $K_{4,4}$ is shown below.

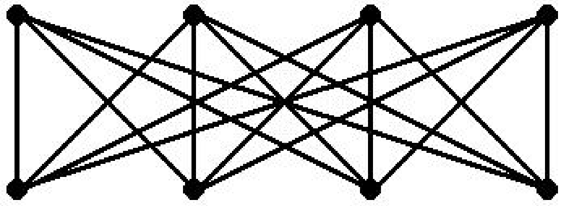


Figure 10: The complete bipartite graph $K_{4,4}$

Imprimitive distance-transitive graphs are characterised in a nice way. This result is due to D.H. Smith [3] in (1971).

Suppose that Γ is a distance-transitive graph, with $\mathrm{diam}(\Gamma) > 2$. Then:

- A block of Γ containing a vertex v must be a union of cells $\Gamma_i(v)$.
- The only possibilities for blocks are
  (a) $\Gamma_0(v) \cup \Gamma_2(v) \cup \Gamma_4(v) \cup \cdots \cup \Gamma_\eta(v)$ (where η is the maximum even distance in Γ, so is either d or $d-1$), or
  (b) $\Gamma_0(v) \cup \Gamma_d(v)$.
- Case (a) occurs if and only if Γ is bipartite.
- Case (b) occurs if and only if Γ is *antipodal*.

Note that Γ is antipodal if, for $u, w \in \Gamma_0(v) \cup \Gamma_d(v)$, $u \neq w$, we have $d(u,w) = d$. Clearly, if $\Gamma_d(v)$ consists only of a single vertex, then Γ must be antipodal, so in this case we call Γ *trivially antipodal*.

- Hence a distance-transitive graph is imprimitive if and only if it is bipartite or antipodal.

Observe that this is an inclusive ‘or’: it is possible for both situations to occur simultaneously. For instance, the cube (see figure 5) is both bipartite and antipodal.

If a graph is trivially antipodal, it is easy to spot (we just have to count the number of vertices in the outermost cell). Bipartite distance-transitive (or, indeed, distance-regular) graphs are also easy to spot: this can be determined immediately from the intersection array.

Proposition
Suppose Γ is distance-regular, with intersection array

$$\iota(\Gamma) = \left\{ \begin{array}{ccccc} * & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & * \end{array} \right\}.$$

Then Γ is bipartite if and only if $a_0 = a_1 = a_2 = \cdots = a_d = 0$.

Proof:
We prove the contrapositive in both directions.

First, suppose $a_j \neq 0$ for some j, so there are adjacent vertices $v, w \in \Gamma_j(u)$. Also, there are paths π, ρ of length j from u to v and from u to w respectively. Let x be the last vertex where π and ρ meet and suppose $d(x,v) = d(x,w) - = m$. Then we can form a circuit of length $2m+1$, an odd number. So Γ is not bipartite.

Conversely, suppose Γ is not bipartite. Then Γ contains some odd circuit σ, of length $2\delta + 1$ say. Choose some vertex u in σ. Then there exist two adjacent vertices v, w in σ such that $d(u,v) = d(u,w) = \delta$, so $v, w \in \Gamma_\delta(u)$ as shown.

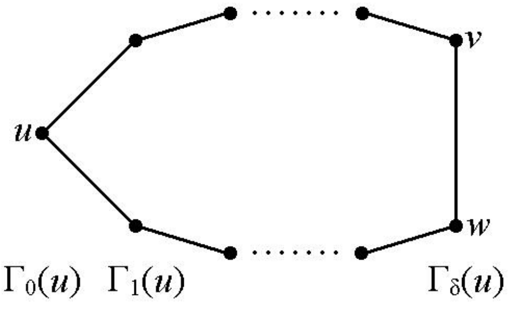


Figure 11: An odd circuit in a non-bipartite graph Γ

Consequently we have that a_δ is non-zero. □

We now return to some of the infinite families of distance-transitive graphs we considered in section 3, and determine if they are primitive.

5.1 Primitive Hamming graphs?

We shall check if these can be bipartite and/or antipodal. Now, the intersection array of $H(d,n)$ is

$$\left\{ \begin{array}{ccccccc} * & 1 & \cdots & j & \cdots & d-1 & d \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & d(n-2) \\ d(n-1) & (d-1)(n-1) & \cdots & (d-j)(n-1) & \cdots & 1(n-1) & * \end{array} \right\}.$$

As mentioned above, this corresponds to a bipartite graph iff the middle row is all zeroes. Now, the columns must sum to the valency, which is $d(n-1)$. So we want

$$\begin{array}{rl} & d(n-1)-j-(d-j)(n-1)=0 \\ \Leftrightarrow & nd-d-j-nd+d+nj-j=0 \\ \Leftrightarrow & nj-2j=0 \\ \Leftrightarrow & n=2. \end{array}$$

Thus the only bipartite Hamming graphs are $H(d,2)$, the hypercubes.

We note that the hypercubes are trivially antipodal. For any distance-regular graph Γ and a fixed vertex v, by counting the number of edges between cells $\Gamma_j(v)$ and $\Gamma_{j+1}(v)$ it follows that $k_j b_j = k_{j+1} c_{j+1}$ (where $k_j = \Gamma_j(v)$). Thus we can obtain a formula for the number of vertices in $\Gamma_d(v)$,

$$k_d = \frac{b_0 b_1 \cdots b_{d-1}}{c_1 c_2 \cdots c_d}.$$

In the case of $H(d,n)$ this gives

$$\begin{array}{rcl} k_d & = & \dfrac{b_0 b_1 \cdots b_{d-1}}{c_1 c_2 \cdots c_d} \\ & = & \dfrac{d(n-1)(d-1)(n-1)\cdots 1(n-1)}{1 \times 2 \times \cdots \times d} \\ & = & \dfrac{d!(n-1)^d}{d!} \\ & = & (n-1)^d. \end{array}$$

Clearly this equals 1 iff $n=2$. If $n>2$, however, we have vertices $u=000\cdots 0$, $v=111\cdots 1$ and $w=211\cdots 1$, where $d(u,v)=d(u,w)=d$, but $d(v,w)=1$. Hence the Hamming graphs $H(d,n)$ for $n>2$ are not antipodal. Thus the only imprimitive Hamming graphs are the hypercubes, so all others must be primitive.

5.2 Primitive Johnson graphs?

We use the same method again, first checking if $J(n,k)$ is bipartite. The intersection array of $J(n,k)$ is given by

$$\left\{ \begin{array}{ccccccc} * & 1^2 & \cdots & i^2 & \cdots & (k-1)^2 & k^2 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & nk-2k^2 \\ k(n-k) & (k-1)(n-k-1) & \cdots & (k-i)(n-k-i) & \cdots & n-2k+1 & * \end{array} \right\}.$$

If the middle row is all zeroes, then for each i we have

$$k(n-k)-i^2-(k-i)(n-k-i)=0$$

and (from the last column) $nk-2k^2=0$. This last condition implies $n=2k$, so substituting this gives us, for all i,

$$\begin{aligned} & k(2k-k)-i^2-(k-i)(2k-k-i)=0 \\ \Leftrightarrow\quad & k^2-i^2-(k-i)^2=0 \\ \Leftrightarrow\quad & 2ki-2i^2=0 \\ \Leftrightarrow\quad & i=k, \end{aligned}$$

which is absurd. So $J(n,k)$ can never be bipartite.

We now observe that if $n=2k$, $J(n,k)$ is trivially antipodal. If vertices $u=\{u_1,u_2,\ldots,u_k\}$ and $v=\{v_1,v_2,\ldots,v_k\}$ are at maximum distance (which clearly is k), we have $u_i\neq v_j$ for all i,j. Therefore these form a partition of the underlying set, so there can be no other vertex at distance k from u. However, if $n>2k$ (we can assume this, as there is an isomorphism from $J(n,k)$ to $J(n,n-k)$, by taking the complement of each k-subset) we have vertices $u=\{u_1,u_2,\ldots,u_k\}$, $v=\{v_1,v_2,\ldots,v_k\}$ and $w=\{v_1,v_2,\ldots,v_{k-1},x\}$, where $x\neq u_i,v_j$ for all i,j. Hence $d(u,v)=d(u,w)=k$, but $d(v,w)=1$. So the only imprimitive Johnson graphs are $J(2k,k)$.

We conclude these notes by mentioning that there are only four primitive distance transitive graphs of valency 3. These are the complete graph K_4, the Petersen graph, the Coxeter graph and the Biggs-Smith graph. The last three of these were of sufficient interest that Norman Biggs described them as “Three Remarkable Graphs” in his paper [2], where full descriptions can be found. We have already seen the Petersen graph in figure 1. The Coxeter graph and Biggs-Smith graph are shown below.

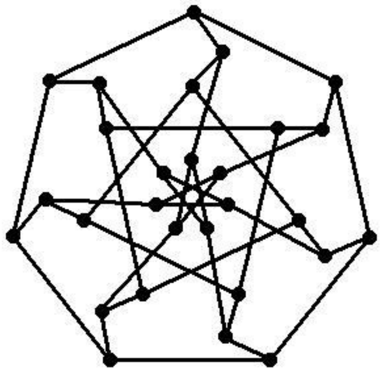


Figure 12: The Coxeter graph

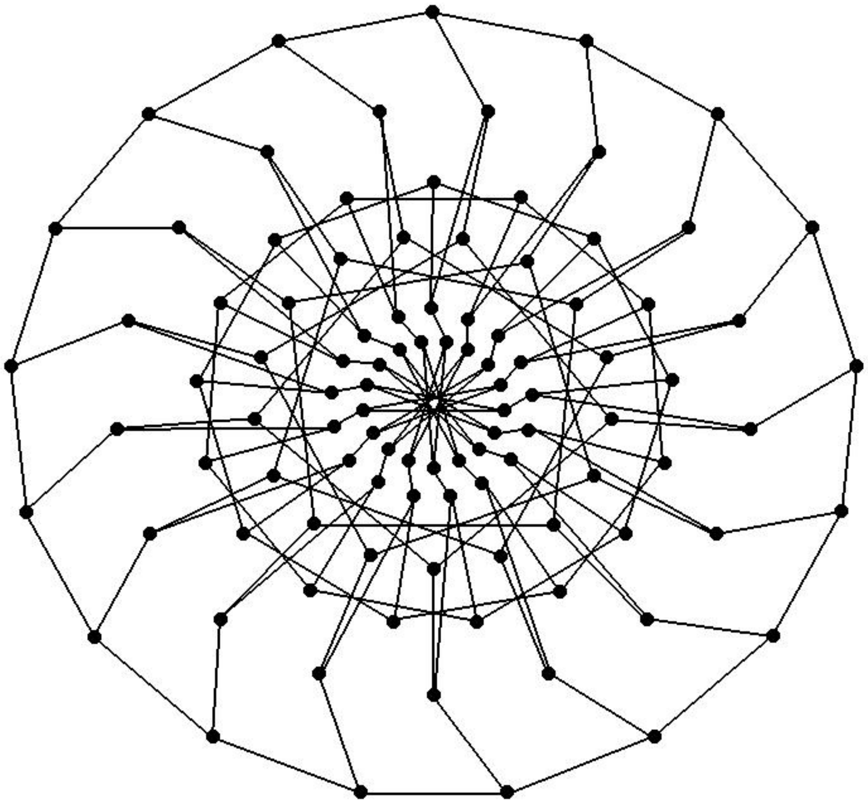


Figure 13: The Biggs-Smith graph

References

[1] R.F. Bailey (2002), *Distance-Transitive Graphs*, M.Math. project, University of Leeds. Available from

`http://www.maths.qmul.ac.uk/∼rfb/dtg.pdf`.

[2] N.L. Biggs (1973), Three remarkable graphs, *Canad. J. Math.* **25**, 397-411.

[3] D.H. Smith (1971), Primitive and imprimitive graphs, *Quart. J. Math. Oxford* (2) **22**, 551-557.