

# The metric dimension problem in geometry, algebra and combinatorics

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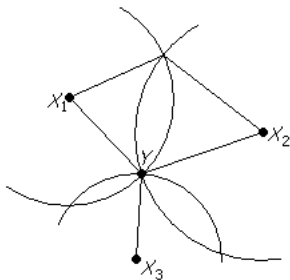
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- ▶ How many friends do you need to figure out where you are?

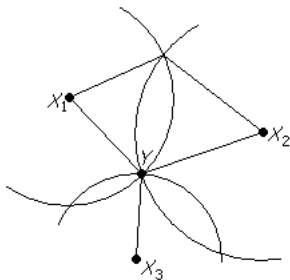
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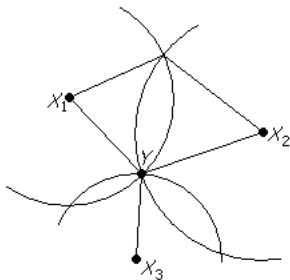
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- ▶ In three dimensions, we would need four friends.

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- ▶ So metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).

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- ▶ It was introduced in the 1970s by Harary and Melter, and (independently) by Slater.
- ▶ Metric dimension of graphs has a variety of applications, including:
  - ▶ sonar
  - ▶ robot navigation
  - ▶ optimization
  - ▶ pharmaceutical chemistry

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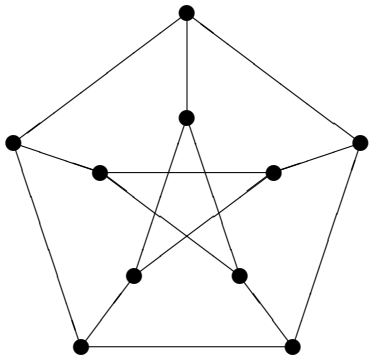
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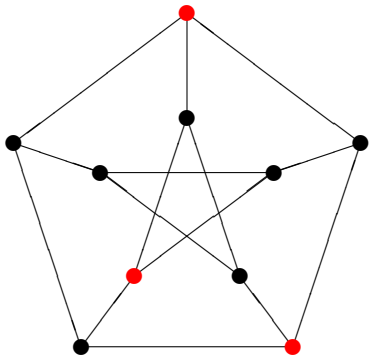
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- ▶ Petersen graph:  $\mu(P) = 3$ .

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- ▶ Also, a graph has metric dimension  $n - 1$  iff it is complete, so for non-complete graphs  $\Gamma$ ,

$$\mu(\Gamma) \leq n - 2.$$

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- ▶ Standard example: the orbitals of a permutation group  $G$  acting on  $V$  form a coherent configuration. (Denote this by  $\mathcal{R}(G)$ .)
- ▶ Another important class of examples are the distance-regular graphs.

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- ▶ Babai used this result to obtain ground-breaking results bounding the possible orders of primitive permutation groups.....

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- ▶ This was how Babai obtained his bounds.

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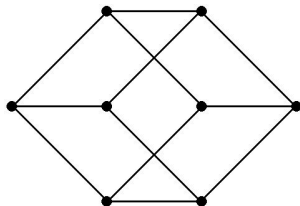
- ▶ Thus we have a bound on  $\mu(\Gamma)$  in terms of its parameters.
- ▶ See the following survey paper for details:  
R. F. Bailey and P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, *Bull. London Math. Soc.* **43** (2011), 209–242.

# Hypercubes

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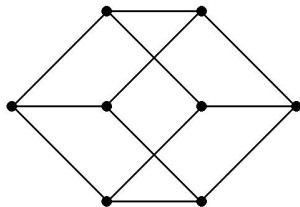
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- ▶ Erdős/Rényi, Lindström (1960s); Sebő/Tannier (2004): for the hypercube  $H(m, 2)$ ,

$$\mu(H(m, 2)) = \frac{2m}{\log_2 m} (1 + o(1)).$$

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- ▶ Chvátal was actually studying strategies for the game *Mastermind*!

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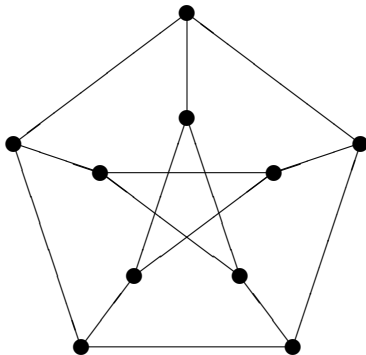
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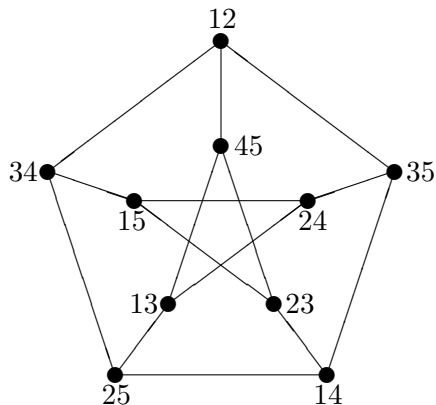
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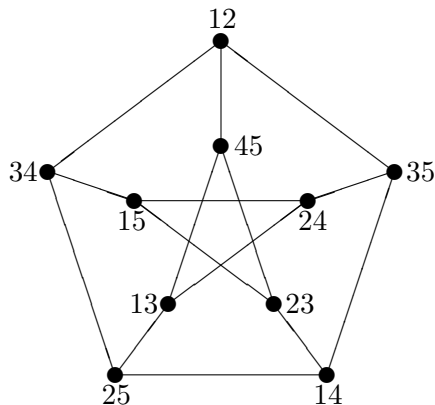
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The Johnson graph  $J(5, 2)$  is its complement.

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- ▶ Notice that these agree when  $k = 2$ : in that case, the bound is sharp (B+Cameron).

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- ▶ For  $K(n, k)$ , do the same, but cut into pieces of size  $2k - 1$  instead.

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- ▶ For Johnson graphs, we also have a different approach.
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$$v_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise.} \end{cases}$$

- ▶ The *incidence matrix* of a family of subsets  $\mathcal{S} = \{X_1, \dots, X_m\}$  is the matrix whose rows are the incidence vectors of  $X_1, \dots, X_m$ .

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- ▶ Notice that  $|U \cap W|$  can be calculated by the dot product of their incidence vectors.

## Johnson graphs and incidence matrices, III

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- ▶ This gives a bound on the metric dimension of  $J(n, k)$ ,

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  - ▶ any pair of points lie on exactly  $\lambda$  blocks;
  - ▶ any pair of blocks agree in exactly  $\lambda$  points.
- ▶ When  $\lambda = 1$ , we have a projective plane, where there are  $q^2 + q + 1$  points and  $q^2 + q + 1$  lines (=blocks), and each line contains  $q + 1$  points (for some integer  $q$ , the *order* of the plane).

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- ▶ Example: the Fano plane can be used as a resolving set of size 7 for  $J(7, 3)$ .

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- ▶ Thus when  $k = 2m - 1$  is odd, Hadamard designs yield resolving sets for odd graphs.

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- ▶ These are the “ $q$ -analogues” of the Johnson graphs.
- ▶ As with Johnson graphs, we may assume that  $2 \leq k \leq n/2$ .
- ▶ Methods for Johnson graphs can easily be adapted.

## Grassmann graphs, II

► **Theorem (B+Meagher):**

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- By a strange piece of numerology, the number of subspaces works out to be the same.....

## Bilinear forms graphs

- ▶ The *bilinear forms graph*  $H_q(n, d)$  is a subgraph of  $G_q(n + d, d)$ , where there is a fixed  $n$ -dimensional subspace  $N$ , and the vertices are those subspaces which intersect  $N$  trivially.

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- ▶ Their proof follows the same method as ours for the Grassmann graphs.

## References

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