Metric dimension of distance-regular graphs

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- Metric dimension was introduced in the 1970s by Harary and Melter, and (independently) by Slater.

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- Petersen graph: $\mu(P) = 3$.

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  2. $\mathcal{R}$ is closed under taking transposes;
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- An important class of coherent configurations are the distance-regular graphs.

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- His most general bound is

$$\mu(\mathcal{R}) < 4\sqrt{n}\log n$$

(where $n = |V|$, and $\mathcal{R}$ is a primitive c.c. of rank at least 3).

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- **Theorem:** Let Γ be a primitive distance-regular graph on n vertices of diameter d. Then the metric dimension of Γ satisfies
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- Thus we have a bound on $\mu(\Gamma)$ in terms of its parameters.

The Hamming scheme

- Erdős/Rényi, Lindström (1960s); Sebő/Tannier (2004): for the hypercube $H(d, 2)$,

$$\lim_{d\to\infty} \frac{\mu(H(d,2)) \log_2 d}{d} = 2.$$

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- Chvátal (1983): where $d > d_\varepsilon$ and $q < d^{1-\varepsilon}$,

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- Cáceres *et al* (2007): for the square lattice graph (or rook's graph) $H(2,q)$,

$$\mu(H(2,q)) = \left\lfloor \frac{2}{3}(2q-1) \right\rfloor.$$

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  - In $J(n,2)$, subsets are adjacent iff they intersect.
  - In $K(n,2)$, subsets are adjacent iff they are disjoint.
- **Theorem:** (B., Cameron) For both $J(n,2)$ and $K(n,2)$, we have

$$\mu(J(n,2)) = \begin{cases} \frac{2}{3}n & n \equiv 0 \pmod 3 \\ \frac{2}{3}(n-1)+1 & n \equiv 1 \pmod 3 \\ \frac{2}{3}(n-2)+2 & n \equiv 2 \pmod 3 \end{cases}.$$

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- Proof (when $k + 1$ divides n):
  - Partition $\{1, \ldots, n\}$ into $(k + 1)$-sets.
  - In each part, take all (but one) of its k-subsets: these form a resolving set.

The Grassmann scheme

Oops!!! The result on this slide was wrong. However, it has since been fixed.....

THE END