Generalized covering designs as hypergraph covers

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Joint work with A. Burgess, M. Cavers, K. Meagher

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An invitation

Two conferences at the University of Regina this summer:

- *Graphs, Designs and Algebraic Combinatorics*
  18–21 July 2011
  `www.math.uregina.ca/~gdac2011`

- *Prairie Discrete Math Workshop 2011*
  22–23 July 2011
  `www.math.uregina.ca/~pdmw2011`

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- The size of the smallest possible (v, k, t) covering design is denoted by $C(v, k, t)$.

Covering designs: an example

- An $(8, 5, 2)$ covering design:

$$
\begin{array}{c}
1\,2\,3\,4\,5 \\
1\,5\,6\,7\,8 \\
2\,3\,6\,7\,8 \\
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- Now let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be an m-tuple of pairwise disjoint sets, where $|X_i| = v_i$.
- An m-tuple $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ of *non-negative* integers is called *admissible*, if they sum to t and each $t_i \leq k_i$.
- Similarly, an m-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_m)$ of disjoint sets is called *admissible*, if each $T_i \subseteq X_i$ and $|T_i| = t_i$, where $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ is admissible.

Generalized covering designs: definition

- A *generalized covering design* $\mathrm{GC}_\lambda(\mathbf{v}, \mathbf{k}, t)$ is a collection of blocks

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- Usually, we are only concerned with the case $\lambda = 1$, and omit the subscript λ.
- The *generalized t-designs* of Cameron (2009) are defined similarly, but strengthened to require "exactly λ" blocks rather than "at least λ".

Generalized covering designs: motivating examples

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- A covering array $\mathrm{CA}(N; k, s, t)$ is a $\mathrm{GC}(\mathbf{v}, \mathbf{k}, t)$, where $\mathbf{v} = (s, s, \ldots, s)$ and $\mathbf{k} = (1, 1, \ldots, 1)$ (both vectors of length k), with N blocks.

Generalized covering designs: motivating examples

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- A covering array $\mathrm{CA}(N;k,s,t)$ is a $\mathrm{GC}(\mathbf{v},\mathbf{k},t)$, where $\mathbf{v}=(s,s,\ldots,s)$ and $\mathbf{k}=(1,1,\ldots,1)$ (both vectors of length k), with N blocks.
- Both t-designs and orthogonal arrays appear as motivating examples for Cameron's generalized t-designs in the same way.

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- The following is a $\mathrm{GC}(\mathbf{v}, \mathbf{k}, 2)$:

$$
\begin{array}{lll}
(\{12\}, & \{1\}, & \{1\}) \\
(\{13\}, & \{1\}, & \{2\}) \\
(\{14\}, & \{2\}, & \{1\}) \\
(\{23\}, & \{2\}, & \{2\}) \\
(\{24\}, & \{1\}, & \{2\}) \\
(\{34\}, & \{2\}, & \{1\})
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- Note that the edges of G correspond precisely to the $(\mathbf{v}, \mathbf{k})$-admissible vectors $\mathbf{T}$.

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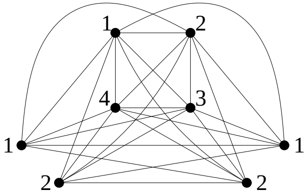
The example of a $\mathrm{GC}((4,2,2),(2,1,1),2)$ we saw earlier can be represented as a clique covering:



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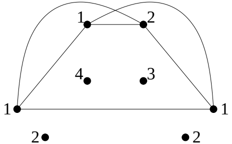
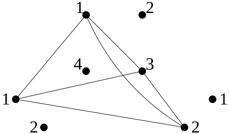


$(\{12\},\ \{1\},\ \{1\})$



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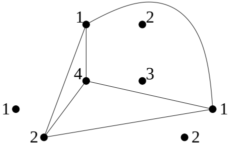

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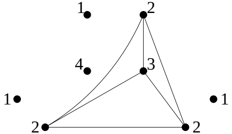

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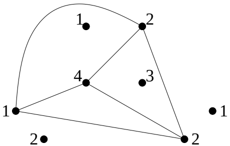

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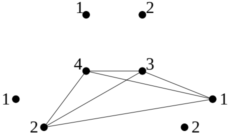

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What if $t > 2$?

- In our paper:

  Bailey, Burgess, Cavers, Meagher, *Generalized covering designs and clique coverings*, J. Comb. Des., to appear; `arXiv.org/abs/1011.3804`

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- To do something similar for larger t, we would need a "hypergraph cover" model instead.

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- Then a $\mathrm{GC}(\mathbf{v}, \mathbf{k}, t)$ corresponds to a clique covering of $\mathcal{H}$ by copies of a complete hypergraph $\mathcal{K}_k^{(t)}$, where each clique contains k_i vertices in X_i for all i.

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- Note that if $t = 2$, we recover our earlier model.

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$$C(\mathbf{v}, \mathbf{k}, t) \geq \max_{\{i_1, \ldots i_t\} \subseteq \{1, \ldots, m\}} \left\lceil \frac{v_{i_1}}{k_{i_1}} \left\lceil \frac{v_{i_2}}{k_{i_2}} \cdots \left\lceil \frac{v_{i_t}}{k_{i_t}} \right\rceil \cdots \right\rceil \right\rceil .$$

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- This is obtained by considering admissible vectors $\mathbf{T}$ with one vertex in each part; the proof follows that of the *Schönheim bound* for ordinary covering designs.

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- Are there situations where they just don't work?
- Any volunteers?

To Be Continued....