A matrix method for resolving sets in Johnson graphs

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An invitation

Two conferences at the University of Regina this summer:

- *Graphs, Designs and Algebraic Combinatorics*
  18–21 July 2011
  `www.math.uregina.ca/~gdac2011`

- *Prairie Discrete Math Workshop 2011*
  22–23 July 2011
  `www.math.uregina.ca/~pdmw2011`

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- Metric dimension was introduced in the 1970s by Harary and Melter, and (independently) by Slater.

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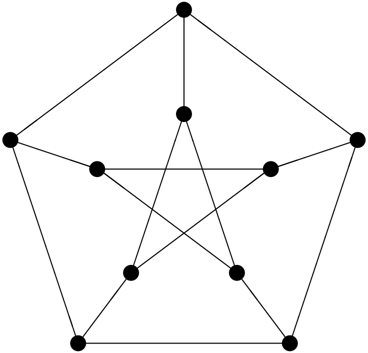
- Trees: a precise formula due to Slater.
- Petersen graph: $\mu(P) = 3$.

Example: Petersen graph



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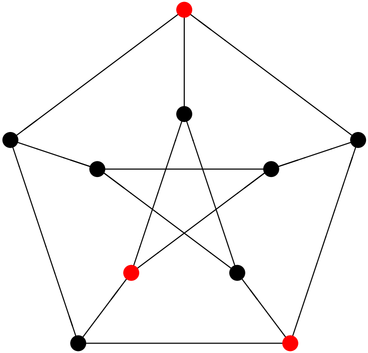

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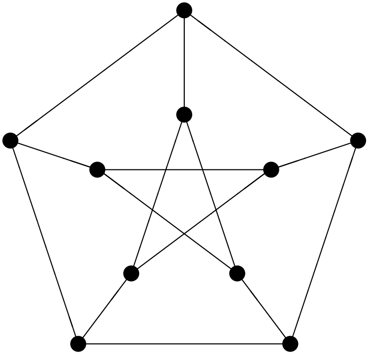
- The *Kneser graph* $K(n, k)$: join two k-subsets if they are disjoint.
- Note that $K(n, k)$ is connected iff $k < \frac{n}{2}$, so we always assume this.

The Petersen graph again

Example: the Kneser graph $K(5, 2)$ is the Petersen graph:



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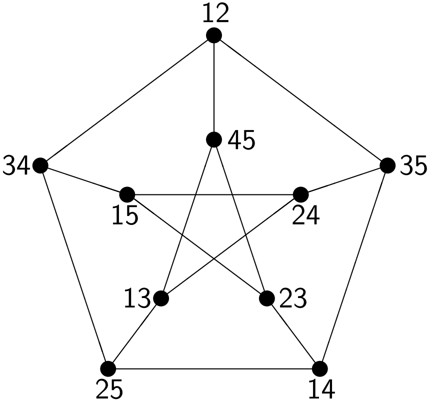

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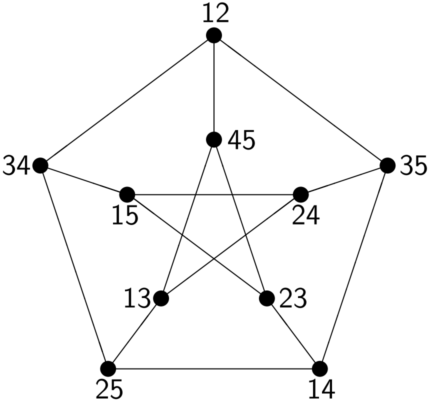

The Johnson graph $J(5,2)$ is its complement.

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  is unique.
- This equivalence (between distances and intersections) does not happen in Kneser graphs, apart from $K(n, 2)$ and $K(2k + 1, k)$.

Incidence matrices

- The *incidence vector* of a k-subset $X \subseteq \{1, \ldots, n\}$ is the 0/1 vector $v = (v_1, \ldots, v_n)$, where

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- The *incidence matrix* of a family of subsets $\mathcal{S} = \{X_1, \ldots, X_m\}$ is the matrix whose rows are the incidence vectors of $X_1, \ldots, X_m$.

Incidence matrices: example

1 2 4
2 3 5
3 4 6
4 5 7
5 6 1
6 7 2
7 1 3

Incidence matrices: example

$$
\begin{array}{l}
124 \\
235 \\
346 \\
457 \\
561 \\
672 \\
713
\end{array}
\qquad
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
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- In particular, if M is an invertible $n \times n$ matrix, $\mathcal{S}$ is a resolving set for $J(n, k)$.
- This gives a bound on the metric dimension of $J(n, k)$,

$$\mu(J(n, k)) \leq n.$$

Johnson graphs and incidence matrices, II

The following matrix gives a resolving set of size n for $J(n, k)$:

$$
\left[\begin{array}{ccccc|cccc}
0 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\
 & & \ddots & & & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 & 1 & & & \vdots & \\
1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\hline
1 & \cdots & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 0 & 0 & 0 & 1 & \cdots & 0 \\
 & \vdots & & & & & & \ddots & \\
1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right]
$$

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- A *symmetric design* is a collection of *points* and *blocks*, with the property that:
  - any pair of points lie in exactly λ blocks;
  - any pair of blocks agree in exactly λ points.
- If $\lambda = 1$, we have a *finite projective plane*. (We call the blocks *lines* in that case.)

Projective planes as resolving sets

- In a finite projective plane, there are $q^2 + q + 1$ points and $q^2 + q + 1$ lines, and each line contains $q + 1$ points (for some integer q, the *order* of the plane).

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- So, if there exists a projective plane of order q, we can use it as a resolving set for $J(q^2 + q + 1, q + 1)$.
- Example: the Fano plane can be used as a resolving set of size 7 for $J(7, 3)$.

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- Another example: if there exists a $4m \times 4m$ Hadamard matrix, one can construct a symmetric design with $4m - 1$ points, blocks of size $2m - 1$, and with $\lambda = m - 1$.
- This forms a resolving set for $J(4m - 1, 2m - 1)$.

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- In particular, when $k = 2m-1$ is odd, the symmetric designs from Hadamard matrices can be used as resolving sets.

THE END