

Hamiltonian decompositions of complete k -uniform hypergraphs

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Hypergraphs

- A *hypergraph* \mathcal{H} is a pair (V, E) , where V is a set of n vertices, and E is a family of subsets of V , called (hyper)edges.
- \mathcal{H} is *k-uniform* if all the hyperedges have size k .
- So a 2-uniform hypergraph is just a graph.
- The *complete k-uniform hypergraph*, denoted $K_n^{(k)}$, has all the k -subsets of the n -set V as its hyperedges.

Hamiltonian circuits

- Katona and Kierstead (1999) gave the following definition:

A Hamiltonian circuit in a k -uniform hypergraph \mathcal{H} is a cyclic ordering of the vertices of \mathcal{H} such that every k consecutive vertices form an edge of \mathcal{H} .

- This generalises the notion of a Hamiltonian circuit in a graph in a sensible way.
- There are also other notions of hamiltonicity in hypergraphs, but we won't discuss those.

Hamiltonian decompositions

- In the case of graphs, the following idea is well-known:

A Hamiltonian decomposition of a graph G is a partition of the set of edges of G into Hamiltonian cycles.

- For complete graphs K_n (where n is odd) Hamiltonian decompositions were discovered in the 1890s by Walecki. Walecki also showed what to do when n is even.

- The following definition therefore seems natural:

A Hamiltonian decomposition of a k -uniform hypergraph \mathcal{H} is a partition of the set of hyperedges of \mathcal{H} into Hamiltonian cycles.

Hamiltonian decompositions, II

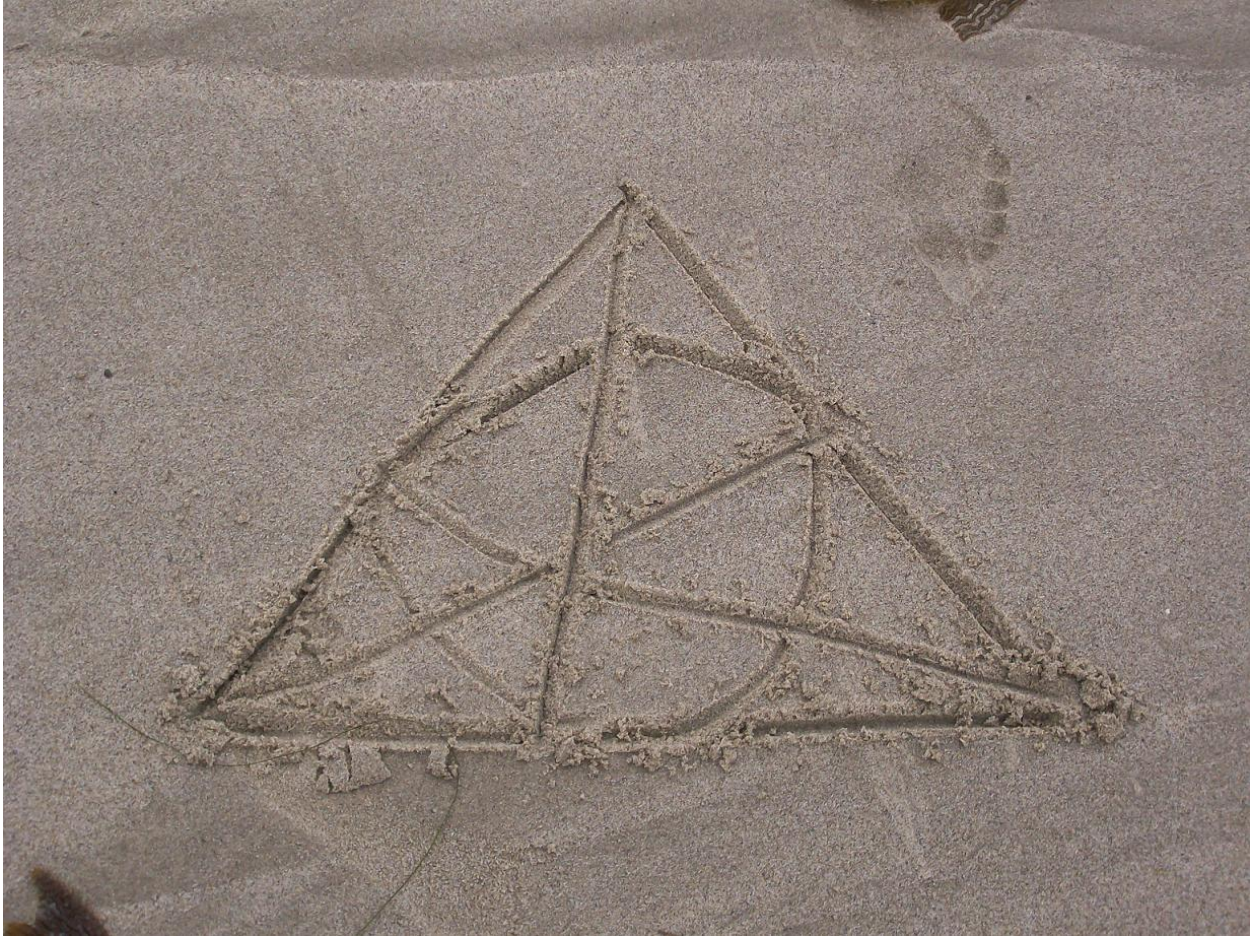
- The natural question which arises is, when does the complete k -uniform hypergraph $K_n^{(k)}$ have a Hamiltonian decomposition?
- An obvious necessary condition is that the number of edges in a circuit, n , must divide the total number of edges, $\binom{n}{k}$.
- We also need that k , the number of edges in a circuit containing a particular vertex, must divide the total number of edges containing that vertex, $\binom{n-1}{k-1}$, but this is equivalent to the last condition.
- We conjecture that these necessary conditions are sufficient (i.e. that $K_n^{(k)}$ has a Hamiltonian decomposition whenever n divides $\binom{n}{k}$).

Computer searches

- To search for Hamiltonian decompositions of $K_n^{(k)}$, we can use the following approach:
 1. Make a graph Γ whose vertices are all the Hamiltonian circuits of $K_n^{(k)}$, and join two circuits if they have no hyperedge in common.
 2. Find a maximum clique inside Γ . If it has size $\binom{n}{k}/n$, then we have found a Hamiltonian decomposition of $K_n^{(k)}$.
- We can use computer packages such as GRAPE to do this.
- Unfortunately for us, this approach runs out of steam very quickly, as the computer runs out of memory just storing the graph.....

t-designs

- A t - (v, k, λ) design consists of a set X of size v (called *points*) together with a family \mathcal{B} of k -subsets of X (called *blocks*), such that any t -subset of X is contained in a constant number λ blocks.
- A Hamiltonian circuit is therefore a type of 1 - (n, k, k) design, where vertices \equiv points, hyperedges \equiv blocks, and each 1 -set of vertices is contained in precisely k hyperedges.
- A *large set* of t - (v, k, λ) designs is a partition of the “complete design” (i.e. $K_n^{(k)}$) into such t -designs.
- Thus a Hamiltonian decomposition is a large set of 1 -designs of this specific type.



A 1-(7,3,3) design

Large sets

- Hartman (1987) showed that for all cases where the necessary numerical conditions are satisfied, there exist large sets of $1-(v, k, \lambda)$ designs.
- His result is proved as a corollary to *Baranyai's Theorem*, which asserts the existence of a partition of $K_n^{(k)}$ into 1-factors (i.e. perfect matchings).
- Unfortunately for us, it is entirely an existence proof, and doesn't give any construction (or even say if the designs in the large set can be isomorphic).
- Note that it has been known since the 19th Century that a large set of Fano planes does not exist, but our computer search shows that there *does* exist a Hamiltonian decomposition of $K_7^{(3)}$.

Difference patterns

- Let $T = \{x_1, x_2, x_3\}$ be an ordered 3-subset of \mathbb{Z}_n . Its *difference pattern* is the set $(x_2 - x_1, x_3 - x_2, x_1 - x_3)$.
- We call a cyclic ordering of \mathbb{Z}_n *multifarious* if all its difference patterns are distinct.
- Example when $n = 10$:

0 1 8 4 6 5 2 7 9 3

- If we take the additive translate of a multifarious ordering, that is also multifarious, with the same set of difference patterns, and each uses a different set of triples.
- We can use multifarious orderings to help search for Hamiltonian decompositions of $K_n^{(3)}$.

Summary of what we've found

Parameters (n, k) for which Hamiltonian decompositions are known are given below:

n/k	2	3	4	5	6	7	8	9	10	11	12	13	14
5	W	W											
6	-	-	-										
7	W	C	C	W									
8	-	C	-	C	-								
9	W	-	C	C	-	W							
10	-	D	?	-	?	D	-						
11	W	D	?	?	?	?	D	W					
12	-	-	-	?	?	?	-	-	-				
13	W	?	?	?	?	?	?	?	?	W			
14	-	?	-	?	-	-	-	?	-	?	-		
15	W	-	?	-	-	?	?	-	-	?	-	W	
16	-	D	-	?	-	?	-	?	-	?	-	D	-

(W=Walecki, C=clique finding, D=difference patterns)