Resolving sets for incidence graphs

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Ottawa–Carleton Discrete Math Days
13 May 2011
[REVISED]

An invitation

Two conferences at the University of Regina this summer:

- *Graphs, Designs and Algebraic Combinatorics*
  18–21 July 2011
  `www.math.uregina.ca/~gdac2011`
- *Prairie Discrete Math Workshop 2011*
  22–23 July 2011
  `www.math.uregina.ca/~pdmw2011`

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- Metric dimension was introduced in the 1970s by Harary and Melter, and (independently) by Slater.

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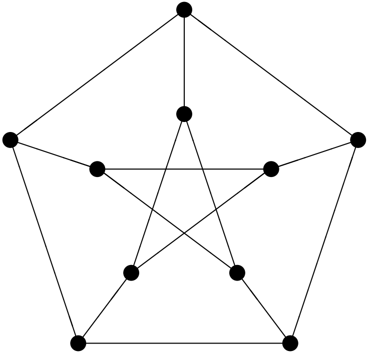
- Petersen graph: $\mu(P) = 3$.

Example: Petersen graph



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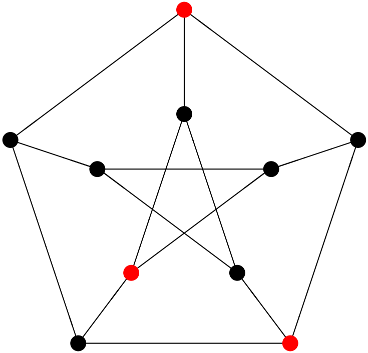

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- If $\lambda = 1$, we have a *finite projective plane*. (We call the blocks *lines* in that case.)

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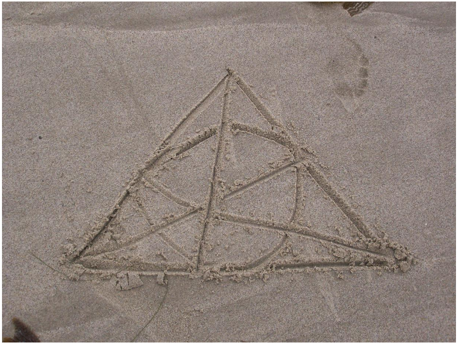

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- This is the *incidence graph* (or *Levi graph*) of the design.

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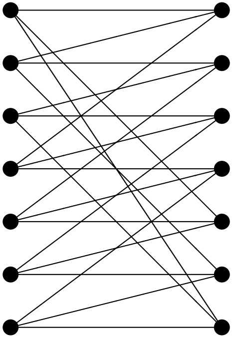

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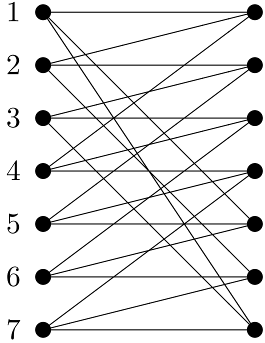

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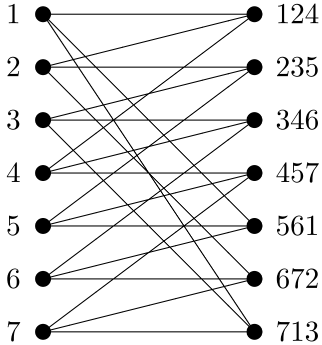

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- From now on, we'll restrict ourselves to $\lambda = 1$, i.e. projective planes.

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- Note that, since any pair of points are at distance 2, if $S \subseteq \mathcal{P}$, we must take at least $|\mathcal{P}| - 1 = q^2 + q$ points.
- So, for a smaller resolving set, we'll need to use both points and lines.

Resolving lines with points: CORRECTED

- We call a set of points $S_{\mathcal{P}}$ a *semi-resolving set* for Γ_{Π} if, for any two distinct lines L_1, L_2, there is a point $p \in S_{\mathcal{P}}$ with $\mathsf{d}(p, L_1) \neq \mathsf{d}(p, L_2)$.

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- How can we construct such a set?

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- In fact, we can delete a point from a double blocking set and still have a semi-resolving set.

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- Thus $\mu(\Gamma_{\mathrm{PG}(2,q)}) \leq 4q + 4\sqrt{q} + 2$ (if $q = p^{2e}$).

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- Thus the size of a resolving set for Γ_{Π} *of this type* has size $\gtrsim 4q + 2\sqrt{2q}$.
- In particular, in the case of $\mathrm{PG}(2, p^{2e})$, the upper and lower bounds are very similar!

THE END