Uncoverings-by-bases for graphic matroids

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Matroids

- A *matroid* is a pair $\mathcal{M} = (E, I)$, where E is a finite set and I is a family of subsets of E (called *independent sets*) satisfying the following axioms:

  (i) $\varnothing \in I$;

  (ii) if $A \in I$ and $B \subseteq A$, then $B \in I$;

  (iii) if $A, B \in I$ and $|A| < |B|$, then $\exists x \in B$ such that $A \cup \{x\} \in I$.

- The maximal independent sets are called *bases*.

- By axiom (iii), bases must all have the same size.

Uncoverings-by-bases, graphic matroids

- Let $\mathcal{M} = (E, I)$ be a matroid.

- An *uncovering-by-bases* for $\mathcal{M}$ (or a t-UBB for $\mathcal{M}$) is a set $\mathcal{U}$ of bases for $\mathcal{M}$ such that any t-subset of E is disjoint from at least one base in $\mathcal{U}$.

We'll be considering the following class of matroids:

- Let $G = (V, E)$ be a graph. The *graphic matroid* $M(G)$ has ground set E, and a subset of E is independent iff it contains no cycle.

- If G is connected, then the bases for $M(G)$ are precisely the spanning trees for G.

UBBs for graphic matroids

- The *edge connectivity* $\kappa(G)$ of a connected graph G is the size of the smallest set of edges whose removal disconnects G.

- Thus is fewer that $\kappa(G)$ edges are removed from G, the resulting graph contains a spanning tree.

- So, for $t \leqslant \kappa(G) - 1$, there exists a t-UBB for $M(G)$.

- From now on, assume $t = \kappa(G) - 1$.

Complete bipartite graphs

- Consider the complete bipartite graph $K_{m,n}$, where $n \geqslant m \geqslant 2$, with vertex set $X \dot{\cup} Y$ where $|X| = m$ and $|Y| = n$.

- $\kappa(K_{m,n}) = \min\{m,n\} = m$, so let $t = m - 1$.

- Let A be an arbitrary t-set of edges. Since $|A| < m$, there exists $u \in X$ incident with no edge of A. Similarly, since $|A| < n$, there exists $v \in Y$ incident with no edge of A.

- Construct a spanning tree using uv and all other edges incident with each of u and v.

- The set of all such spanning trees forms a t-UBB for $M(K_{m,n})$.

Complete graphs (odd order)

- Consider the complete graph K_n, where n is odd (set $n = 2k+1$).

- $\kappa(K_n) = n-1 = 2k$, so let $t = n-2 = 2k-1$.

- Again, let A denote an arbitrary t-set of edges.

- Idea: build a t-UBB where the spanning trees are paths.

- Why? They are contained inside Hamilton cycles, and it follows from Ore's Theorem that $K_n \setminus A$ is Hamiltonian.

Complete graphs (odd order)

- Walecki (1892) showed that K_n ($n = 2k+1$) has a decomposition into k disjoint Hamilton cycles.

- Let $\mathcal{D} = \{C_1, \ldots, C_k\}$ be such a decomposition, and for each $C_i \in \mathcal{D}$, form $n = 2k+1$ paths $C \setminus e$ (for each edge $e \in C$), giving $k(2k+1)$ paths altogether.

- We claim that this is a t-UBB for $M(K_n)$:

  - ⋆ If $\exists C_i \in D$ with $A \cap C_i = \varnothing$, take any path in C_i.

  - ⋆ If not, then A meets every cycle, so the $2k-1$ edges are spread across all k cycles.
    $\Rightarrow$ $\exists$ cycle C_j containing just one edge $e \in A$.
    $\Rightarrow$ Use the path $C_j \setminus e$.

Graphs with Hamiltonian decompositions

- In fact, this same construction works for *any* graph G with a Hamiltonian decomposition.

- Suppose the decomposition has k Hamilton cycles.

- G must be $2k$-regular, so $\kappa(G) \leqslant 2k$.

- Also, for any edge-cut of G, each of the k Hamilton cycles must cross it at least twice, so $\kappa(G) \geqslant 2k$.

- Hence $\kappa(G) = 2k$, so take $t = 2k - 1$ and the same construction can be used.

Complete graphs (even order)

- Now consider K_n where $n = 2k$ is even.

- $\kappa(G) = n-1 = 2k-1$, so let $t = n-2 = 2k-2$.

- In this case, there is no Hamiltonian decomposition. However, there is a decomposition into $k-1$ Hamilton cycles and a 1-factor.

- Why don't we try using the same construction as before, using this "near-decomposition"?

Complete graphs (even order)

- Because it doesn't quite work!

- For each of the Hamilton cycles $C_1, \ldots, C_{k-1}$, we can construct $n = 2k$ paths. This would be a t-UBB except for one problem case: where the set A of $2k-2$ "bad" edges consists of two edges from each of our $k-1$ cycles.

- Fortunately, this *can* be fixed, by taking some extra cycles which use the edges in the 1-factor.

“Optimally uncoverable” graphs

- Suppose we have a graph G, and want to “uncover” any s edges (for $s \leqslant t$).

- If G happens to contain $s+1$ *edge-disjoint* spanning trees, then these will form an s-UBB for G.

- To help with this, define $\tau(G)$ to be the maximum number of edge-disjoint spanning trees in a G.

- We call G *optimally uncoverable* if $\tau(G) = \kappa(G)$.

k-joins

- The following operation is very useful when studying optimally uncoverable graphs.

- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with disjoint vertex and edge sets.

- The *k-join*, $G_1 *_k G_2$, is the graph G with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup K$, where K is a set of k edges with one end in V_1 and the other in V_2.

- It is easy to see that if $\kappa(G_1) = \tau(G_1) = \kappa(G_2) = \tau(G_2) = k$, then $\kappa(G_1 *_k G_2) = \tau(G_1 *_k G_2) = k$ also.

Optimally uncoverable matroids

- It is straightforward to generalise the “optimally uncoverable” notion to matroids in general.

- We can define analogues of the parameters τ and κ for a matroid:

  - ⋆ $\tau(M)$ is the maximum number of disjoint bases for M;

  - ⋆ $\kappa(M)$ is the size of a *minimal cocircuit* of M.

- Describing such matroids seems harder than for graphs!

Current work

- Properly characterise optimally-uncoverable graphs.

- Find appropriate “joining” operations for matroids.

- Study the optimally-uncoverable matroids, and try to see if a characterisation is feasible.