Hamiltonian decompositions of complete k-uniform hypergraphs

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or

Why research is a complete pain sometimes

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Hypergraphs

- A *hypergraph* $\mathcal{H}$ is a pair (V,E), where V is a set of n vertices, and E is a family of subsets of V, called (hyper)edges.

- $\mathcal{H}$ is k*-uniform* if all the hyperedges have size k.

- So a 2-uniform hypergraph is just a graph.

- The *complete* k*-uniform hypergraph*, denoted $K_n^{(k)}$, has all the k-subsets of the n-set V as its hyperedges.

Hamiltonian circuits

- Katona and Kierstead (1999) gave the following definition:

  A *Hamiltonian circuit* in a k-uniform hypergraph $\mathcal{H}$ is a cyclic ordering of the vertices of $\mathcal{H}$ such that every k consecutive vertices form an edge of $\mathcal{H}$.

- This generalises the notion of a Hamiltonian circuit in a graph in a sensible way.

- There are also other notions of hamiltonicity in hypergraphs, but we won't discuss those.

Hamiltonian decompositions

- In the case of graphs, the following idea is well-known:

  A *Hamiltonian decomposition* of a graph G is a partition of the set of edges of G into Hamiltonian cycles.

- For complete graphs K_n (where n is odd) Hamiltonian decompositions were discovered in the 1890s by Walecki. Walecki also showed what to do when n is even.

- The following definition therefore seems natural:

  A *Hamiltonian decomposition* of a k-uniform hypergraph $\mathcal{H}$ is a partition of the set of hyperedges of $\mathcal{H}$ into Hamiltonian cycles.

Hamiltonian decompositions, II

- The natural question which arises is, when does the complete k-uniform hypergraph $K_n^{(k)}$ have a Hamiltonian decomposition?

- An obvious necessary condition is that the number of edges in a circuit, n, must divide the total number of edges, $\binom{n}{k}$.

- We also need that k, the number of edges in a circuit containing a particular vertex, must divide the total number of edges containing that vertex, $\binom{n-1}{k-1}$, but this is equivalent to the last condition.

- We conjecture that these necessary conditions are sufficient (i.e. that $K_n^{(k)}$ has a Hamiltonian decomposition whenever n divides $\binom{n}{k}$).

Computer searches

- To search for Hamiltonian decompositions of $K_n^{(k)}$, we can use the following approach:

  1. Make a graph Γ whose vertices are all the Hamiltonian circuits of $K_n^{(k)}$, and join two circuits if they have no hyperedge in common.

  2. Find a maximum clique inside Γ. If it has size $\binom{n}{k}/n$, then we have found a Hamiltonian decomposition of $K_n^{(k)}$.

- We can use computer packages such as GRAPE to do this.

- Unfortunately for us, this approach runs out of steam very quickly, as the computer runs out of memory just storing the graph......

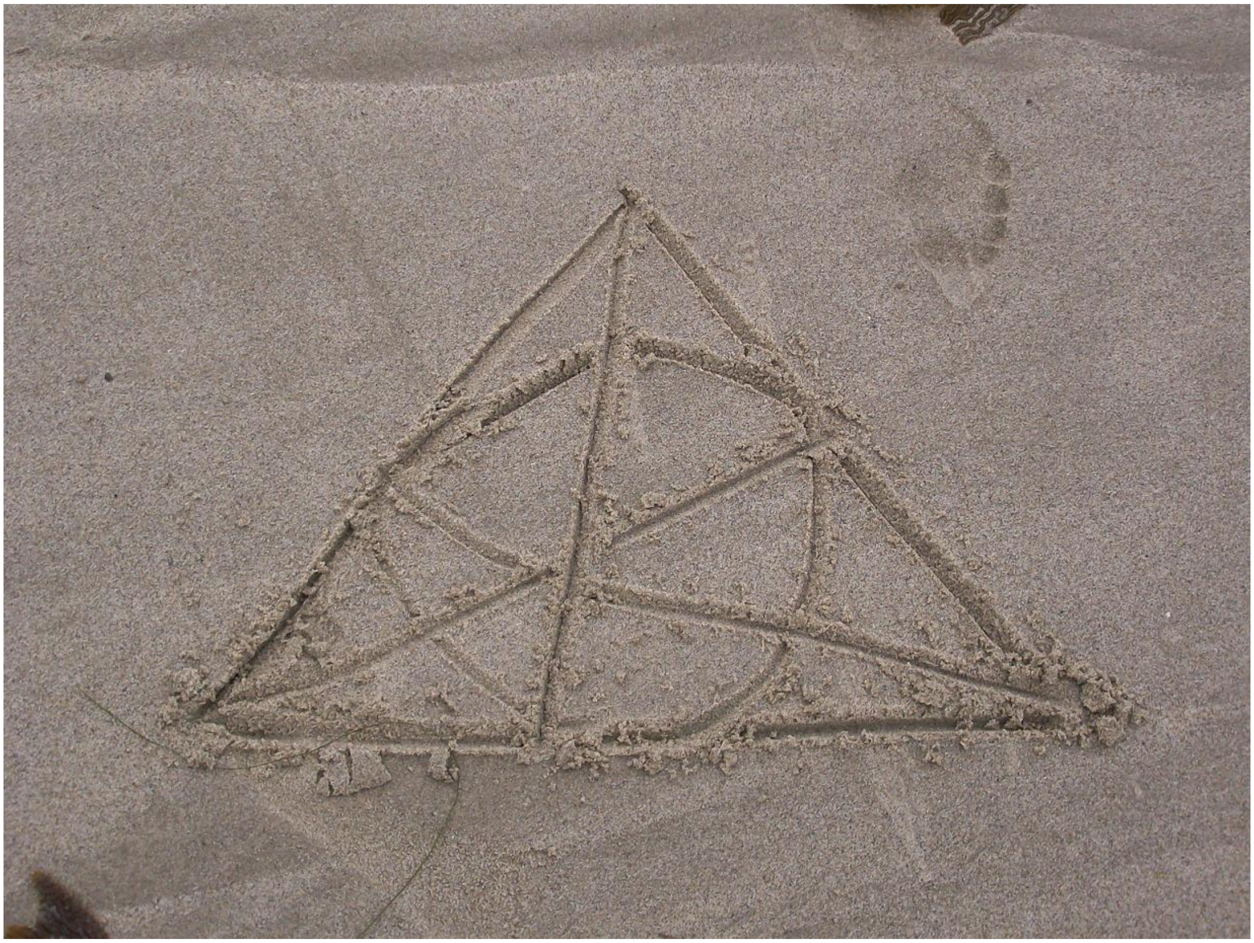
t-designs

- A t-(v,k,λ) design consists of a set X of size v (called *points*) together with a family $\mathcal{B}$ of k-subsets of X (called *blocks*), such that any t-subset of X is contained in a constant number λ blocks.

- A Hamiltonian circuit is therefore a type of 1-(n,k,k) design, where vertices$\equiv$points, hyperedges$\equiv$blocks, and each 1-set of vertices is contained in precisely k hyperedges.

- A *large set* of t-(v,k,λ) designs is a partition of the "complete design" (i.e. $K_n^{(k)}$) into such t-designs.

- Thus a Hamiltonian decomposition is a large set of 1-designs of this specific type.



A 1-$(7,3,3)$ design

Large sets

- Hartman (1987) showed that for all cases where the necessary numerical conditions are satisfied, there exist large sets of 1-(v,k,λ) designs.

- His result is proved as a corollary to *Baranyai's Theorem*, which asserts the existence of a partition of $K_n^{(k)}$ into 1-factors (i.e. perfect matchings).

- Unfortunately for us, it is entirely an existence proof, and doesn't give any construction (or even say if the designs in the large set can be isomorphic).

- Note that it has been known since the 19th Century that a large set of Fano planes does not exist, but our computer search shows that there *does* exist a Hamiltonian decomposition of $K_7^{(3)}$.

Terraces

- Suppose our set of n vertices are in fact $\mathbb{Z}_n$.
- A *terrace* for $\mathbb{Z}_n$ is an ordering of the elements so that each possible difference appears exactly once.
- Example where $n = 12$:

  $$0\ 11\ 1\ 10\ 2\ 9\ 3\ 8\ 4\ 7\ 5\ 6$$

  The differences are:

  $$11\ 2\ 9\ 4\ 7\ 6\ 5\ 8\ 3\ 10\ 1$$

- Although he didn't know it at the time, Walecki's construction utilised terraces.

Difference patterns

- We (sort of) generalise the idea of terraces using the idea of difference patterns.

- Let $T = \{x_1, x_2, x_3\}$ be an ordered 3-subset of $\mathbb{Z}_n$. Its *difference pattern* is the set $\{x_2 - x_1, x_3 - x_2, x_1 - x_3\}$.

- We call a cyclic ordering of $\mathbb{Z}_n$ *3-admissable* if all its difference patterns are distinct.

- Example when $n = 10$:

$$0\ 1\ 8\ 4\ 6\ 5\ 2\ 7\ 9\ 3$$

- If we take the additive translate of a 3-admissable word, that is also 3-admissable, with the same set of difference patterns, and each uses a different set of triples.

- We can use 3-admissable words to help search for Hamiltonian decompositions of $K_n^{(3)}$.

Johnson graphs

- The *Johnson graph* $J(n,k)$ has the edges of $K_n^{(k)}$ as its vertex set, and two vertices are adjacent iff they overlap in a $(k-1)$-set.

- A circuit in the hypergraph $K_n^{(k)}$ gives a cycle in the Johnson graph $J(n,k)$.

- A Hamiltonian decomposition of $K_n^{(k)}$ gives a set of *vertex-disjoint* n-cycles in $J(n,k)$, which cover all of its vertices.

- There are results in graph theory which can be used to show the existence of such things.

- Trouble is, there exist other cycles in $J(n,k)$ which don't correspond to circuits in $K_n^{(k)}$! So, we're stuck again.....

Summary of what we've found

So far, computer searches have solved the following cases:

- $n = 7$, $k = 3$: HD exists (clique-finding)
- $n = 8$, $k = 3$: HD exists (clique-finding)
- $n = 9$, $k = 4$: HD exists (clique-finding)
- $n = 10$, $k = 3$: HD exists (via difference patterns)
- $n = 11$, $k = 3$: HD exists (via difference patterns)
- $n = 13$, $k = 3$ and $n = 14$, $k = 3$: ????
- $n = 16$, $k = 3$: HD exists (via difference patterns)

All others remain open!