The metric dimension problem in geometry, combinatorics and algebra

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Prairie Network for Research in the Mathematical Sciences
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- How many friends do you need to figure out where you are?

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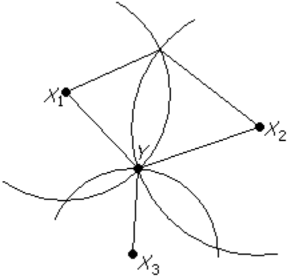
- The answer, of course, is three:



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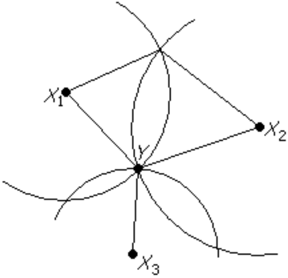

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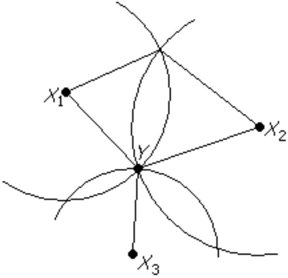

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- In three dimensions, we would need four friends.

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- **Example:** if $X = \mathbb{R}^n$, and $\mathrm{d}(x, y) = ||x - y||$, the metric dimension is $n + 1$ (the *affine dimension* of $\mathbb{R}^n$).
- So metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).

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- Another example is an *error-correcting code*, where d is the *Hamming distance*.

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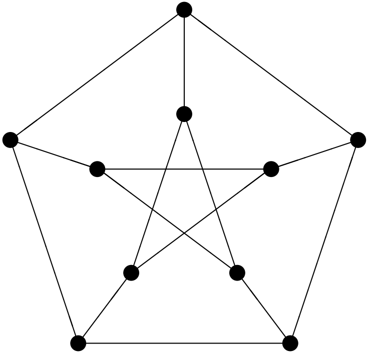
- Petersen graph: $\mu(P) = 3$.

Example: Petersen graph



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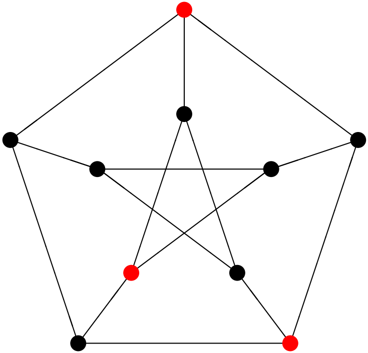

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$$\mu(G) \gtrsim \log_d n.$$
- Also, a graph has metric dimension $n - 1$ iff it is complete, so for non-complete graphs,
$$\mu(G) \leq n - 2.$$

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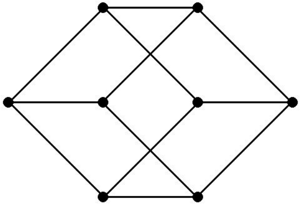

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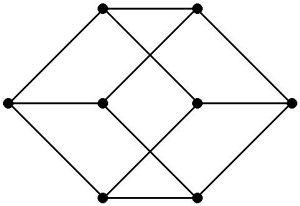

- Erdős/Rényi, Lindström (1960s); Sebő/Tannier (2004): for the hypercube $H(m,2)$,

$$\mu(H(m,2)) = \frac{2m}{\log_2 m}(1+o(1)).$$

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- Chvátal was actually studying strategies for the game *Mastermind*.

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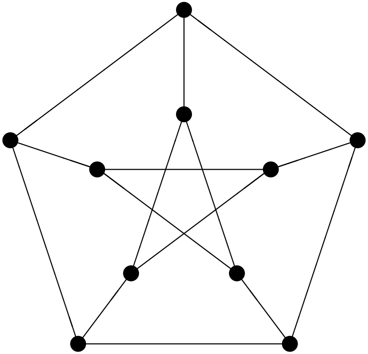
- There are two important ways of making a graph with V as a vertex set.
- The *Johnson graph* $J(n, k)$: join two k-subsets if they intersect in $k - 1$ points.
- The *Kneser graph* $K(n, k)$: join two k-subsets if they are disjoint.

The Petersen graph again

Example: the Kneser graph $K(5,2)$ is the Petersen graph:



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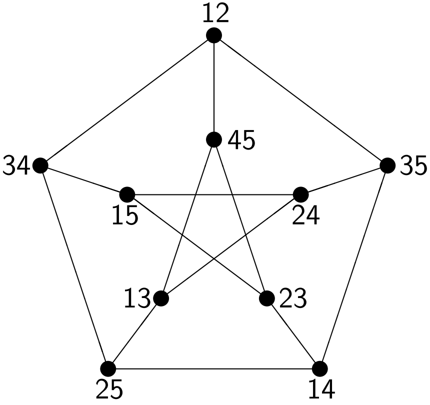

The Johnson graph $J(5,2)$ is its complement.

Johnson and Kneser graphs, II

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- For $K(n, k)$, do the same, but cut into pieces of size $2k - 1$ instead.

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- The *incidence vector* of a k-subset $X \subseteq \{1, \ldots, n\}$ is the $0/1$ vector $v = (v_1, \ldots, v_n)$, where

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- The *incidence matrix* of a family of subsets $\mathcal{S} = \{X_1, \ldots, X_m\}$ is the matrix whose rows are the incidence vectors of $X_1, \ldots, X_m$.

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- Notice that $|U \cap W|$ can be calculated by the dot product of their incidence vectors.

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- This gives a bound on the metric dimension of $J(n, k)$,

$$\mu(J(n, k)) \leq n.$$

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What combinatorialists do at the beach

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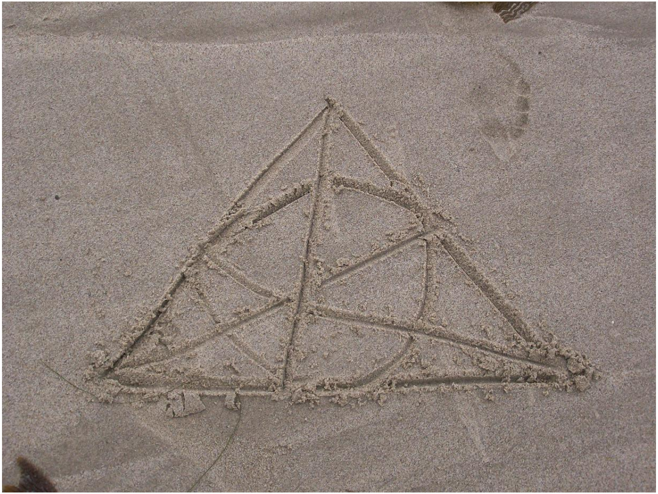

Projective planes as resolving sets

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- So, if there exists a projective plane of order q, we can use it as a resolving set for $J(q^2 + q + 1, q + 1)$.
- Example: the Fano plane can be used as a resolving set of size 7 for $J(7, 3)$.

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- Babai (1981) used a parameter similar to metric dimension to obtain ground-breaking bounds on the orders of finite primitive permutation groups.
- Babai's work gives bounds on metric dimension of distance-regular graphs.
- See the following survey paper for details:
  R. F. Bailey and P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, *Bull. London Math. Soc.* **43** (2011), 209–242.

THE END

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