Generalized Covering Designs

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- Usually, we are only concerned with the case $\lambda = 1$, and omit the subscript λ.
- The size of the smallest possible (v, k, t) covering design is denoted by $C(v, k, t)$.

Covering designs: an example

- An $(8, 5, 2)$ covering design:

$$\begin{array}{c} 1\,2\,3\,4\,5 \\ 1\,5\,6\,7\,8 \\ 2\,3\,6\,7\,8 \\ 4\,5\,6\,7\,8 \end{array}$$

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\begin{array}{c}
1\,2\,3\,4\,5 \\
1\,5\,6\,7\,8 \\
2\,3\,6\,7\,8 \\
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\end{array}
$$

- Each pair chosen from $\{1, \ldots, 8\}$ is contained in at least one of the 5-sets given.

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  - in every set of t columns, each t-tuple occurs in at least λ rows.

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Covering arrays: an example

- A covering array $CA(5; 4, 2, 2)$:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$$

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$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$$

- In each pair of columns, each of the 2^2 possible combinations $00, 01, 10, 11$ appears in at least one row.

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- Question: Is there a similar generalization for covering designs and covering arrays?

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- Several people are now working on these "generalized t-designs".
- Question: Is there a similar generalization for covering designs and covering arrays?
- Answer: Yes.....

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- Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be an m-tuple of positive integers with sum v, and let $\mathbf{k} = (k_1, k_2, \ldots, k_m)$ be an m-tuple of positive integers with sum k, and where each $k_i \leq v_i$.

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- Now let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be an m-tuple of pairwise disjoint sets, where $|X_i| = v_i$.

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- Now let $\mathbf{X} = (X_1, X_2, \ldots, X_m)$ be an m-tuple of pairwise disjoint sets, where $|X_i| = v_i$.
- An m-tuple $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ of *non-negative* integers is called *admissible*, if they sum to t and each $t_i \leq k_i$.
- Similarly, an m-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_m)$ of disjoint sets is called *admissible*, if each $T_i \subseteq X_i$ and $|T_i| = t_i$, where $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ is admissible.

A common generalization: definition

- A *generalized covering design* $GC_\lambda(\mathbf{v}, \mathbf{k}, t)$ is a collection of blocks

$$\mathbf{B} = (B_1, B_2, \ldots, B_m),$$

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- Usually, we are only concerned with the case $\lambda = 1$, and omit the subscript λ.
- Cameron's generalized t-designs are defined similarly, but strengthened to require "exactly λ" blocks rather than "at least λ".

A common generalization: motivating examples

- A (v, k, t) covering design is a $GC(\mathbf{v}, \mathbf{k}, t)$, where $\mathbf{v} = (v)$ and $\mathbf{k} = (k)$.

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- A covering array $CA(N; k, s, t)$ is a $GC(\mathbf{v}, \mathbf{k}, t)$, where $\mathbf{v} = (s, s, \ldots, s)$ and $\mathbf{k} = (1, 1, \ldots, 1)$ (both vectors of length k), with N blocks.

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- So we do indeed have a common generalization!
- Both t-designs and orthogonal arrays appear as motivating examples for Cameron's generalized t-designs in the same way.

A common generalization: another example

- Suppose $\mathbf{v} = (5, 6, 7)$, $\mathbf{k} = (3, 4, 3)$ and $t = 2$.

A common generalization: another example

- Suppose $\mathbf{v} = (5, 6, 7)$, $\mathbf{k} = (3, 4, 3)$ and $t = 2$.
- The following is a $GC(\mathbf{v}, \mathbf{k}, 2)$:

$$
\begin{array}{l}
(\{124\}, \quad \{1234\}, \quad \{124\}) \\
(\{235\}, \quad \{1235\}, \quad \{235\}) \\
(\{134\}, \quad \{1346\}, \quad \{346\}) \\
(\{145\}, \quad \{1245\}, \quad \{457\}) \\
(\{125\}, \quad \{1256\}, \quad \{156\}) \\
(\{123\}, \quad \{1236\}, \quad \{267\}) \\
(\{123\}, \quad \{1234\}, \quad \{137\})
\end{array}
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An algorithm

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  - In any part where $v_i < v_{\max}$, delete the extra points, replacing them with a "placeholder" symbol $\star$.

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  - In any part where $v_i < v_{\max}$, delete the extra points, replacing them with a "placeholder" symbol $\star$.
  - In any part where $k_i > k_{\min}$, add $k_i - k_{\min}$ placeholders to each block.

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  - In each block, replace the placeholders greedily, ensuring that no symbol is repeated in a block.

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  - Remove any repeated blocks.

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  - In each block, replace the placeholders greedily, ensuring that no symbol is repeated in a block.
  - Remove any repeated blocks.
- If $\mathcal{C}$ is optimal, and there is an index i where $v_i = v_{\max}$ and $k_i = k_{\min}$, then we are guaranteed our $GC(\mathbf{v}, \mathbf{k}, 2)$ is also optimal.

That example again

- Suppose $\mathbf{v} = (5, 6, 7)$, $\mathbf{k} = (3, 4, 3)$ and $t = 2$; then $v_{\max} = 7$ and $k_{\min} = 3$.

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- Start with a $(7, 3, 2)$ covering design $\mathcal{C}$:

$$\{124\}$$
$$\{235\}$$
$$\{346\}$$
$$\{457\}$$
$$\{156\}$$
$$\{267\}$$
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- Put a copy of $\mathcal{C}$ on each part:

$$\begin{array}{ccc}
(\{124\}, & \{124\}, & \{124\}) \\
(\{235\}, & \{235\}, & \{235\}) \\
(\{346\}, & \{346\}, & \{346\}) \\
(\{457\}, & \{457\}, & \{457\}) \\
(\{156\}, & \{156\}, & \{156\}) \\
(\{267\}, & \{267\}, & \{267\}) \\
(\{137\}, & \{137\}, & \{137\})
\end{array}$$

That example again

- Suppose $\mathbf{v} = (5, 6, 7)$, $\mathbf{k} = (3, 4, 3)$ and $t = 2$; then $v_{\max} = 7$ and $k_{\min} = 3$.
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\begin{array}{lll}
(\{124\}, & \{124\}, & \{124\}) \\
(\{235\}, & \{235\}, & \{235\}) \\
(\{34\star\}, & \{346\}, & \{346\}) \\
(\{45\star\}, & \{45\star\}, & \{457\}) \\
(\{15\star\}, & \{156\}, & \{156\}) \\
(\{2\star\star\}, & \{26\star\}, & \{267\}) \\
(\{13\star\}, & \{13\star\}, & \{137\})
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(\{45\star\}, & \{45\star\star\}, & \{457\}) \\
(\{15\star\}, & \{156\star\}, & \{156\}) \\
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That example again

- Suppose $\mathbf{v} = (5, 6, 7)$, $\mathbf{k} = (3, 4, 3)$ and $t = 2$; then $v_{\max} = 7$ and $k_{\min} = 3$.
- Replace the placeholders greedily, ensuring that no symbol is repeated in a block:

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\begin{array}{l}
(\{124\}, \quad \{1234\}, \quad \{124\}) \\
(\{235\}, \quad \{1235\}, \quad \{235\}) \\
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