Base size, metric dimension and other stories

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Commandments

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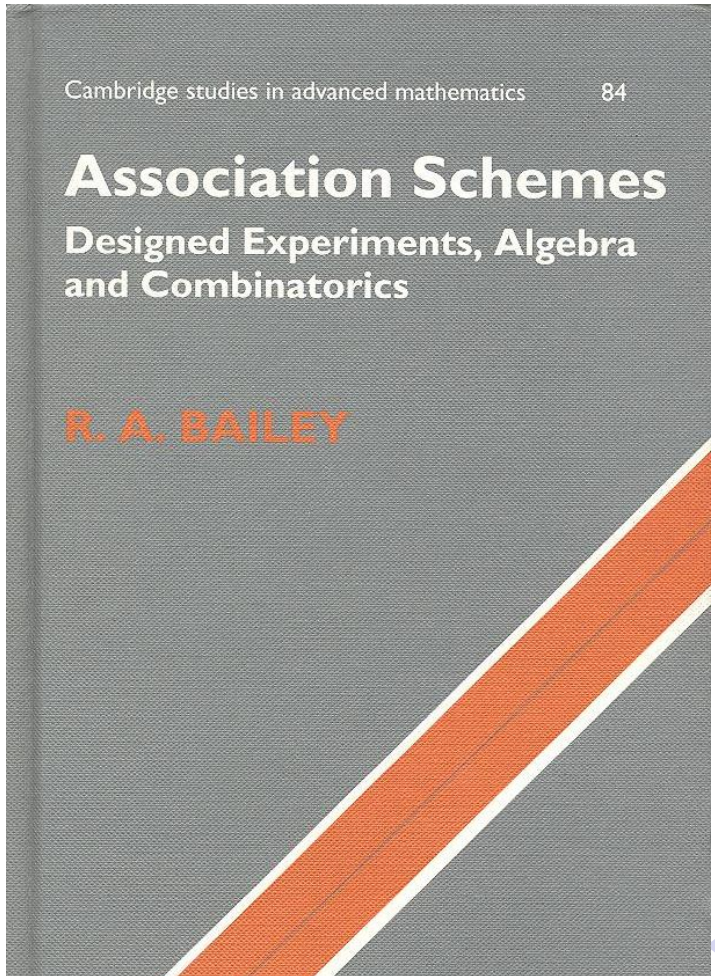
- I am NOT related to the author of this book!



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- Bases were introduced in the 1960s by Sims, in the context of computational group theory.
- The **base size** of G, denoted $b(G)$, is the smallest size of a base for G.

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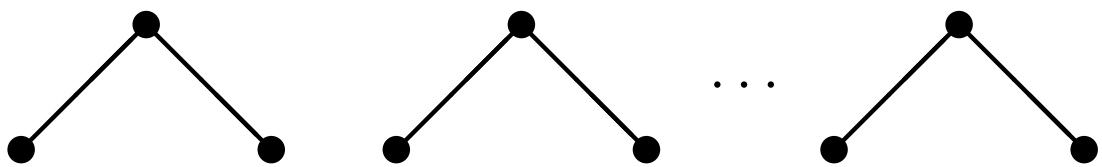
- A base for this action looks like



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- Various authors have worked on this, including Cameron, Kantor, Burness, Liebeck, Shalev, Guralnick, Saxl, James, O'Brien, Wilson, ...
- Most attention has been focused on primitive actions of almost simple groups, where it is shown for "non-standard" actions that $b(G) \leq 7$.

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- Fijavž and Mohar (2004): **rigidity index**
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- In each case, it is regarded as a parameter of the graph, rather than its automorphism group.

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- A subset $\{v_1, v_2, \ldots, v_k\} \subseteq V$ is a **resolving set** for Γ if, for every vertex w, the list of distances

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- Metric dimension was introduced in the 1970s by Harary and Melter, and (independently) by Slater.

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- Petersen graph: $\mu(P) = 3$.

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- *Proof:* Use the fact that $d(u, v) = d(u^g, v^g)$ $(\forall g \in \mathrm{Aut}(\Gamma))$ to show that the pointwise stabiliser is trivial.
- **Corollary:** Where $G \leq \mathrm{Aut}(\Gamma)$, we have $b(G) \leq \mu(\Gamma)$.

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- By bounding $\mu(\mathcal{R})$, Babai was able to bound $b(G)$ and thus $|G|$ (where G is primitive).
- Babai's most general bound is

$$\mu(\mathcal{R}) < 4\sqrt{n}\log n$$

(where $n = |\Omega|$, and $\mathcal{R}$ has rank at least 3).

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- **Theorem:** Let Γ be a distance-regular graph on n vertices of diameter d. Then the metric dimension of Γ satisfies

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- Thus we have a bound on $\mu(\Gamma)$ in terms of its parameters.

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- Since Paley graphs are distance-transitive, the separation index and metric dimension are equal.

The Hamming scheme

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- Let $G = \mathrm{Aut}(H(d,q))$. For all cases where $b(G)$ is known, the difference $\mu(H(d,q)) - b(G)$ can be made arbitrarily large.

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- So this gives an (interesting) infinite family where the two parameters are equal.

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THE END