MATH 103 Problem Set 2 Solutions DRAFT

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Thursday, February 8, 2007

1. (a) $f'(x) = (2/3)x^{-1/3}$

   (b) $f(x) = 1/x^{1/6} = x^{-1/6}$ so $f'(x) = (-1/6)x^{-7/6}$

   (c) f is constant so $f'(x) = 0$.

2. (a) $f(8) = 8^{4/3} = (8^{1/3})^4 = 2^4 = 16$. $f'(x) = (4/3)x^{1/3}$ so $f'(8) = (4/3)8^{1/3} = (4/3)2 = 8/3$.

   (b) The tangent line has slope $f'(8) = 8/3$ and passes through the point $(8, f(8)) = (8, 16)$. By the point-slope form of the line we have $y - 16 = (8/3)(x-8)$. Now in order to answer the question (find a and b) we must simplify at this point. We have $y - 16 = (8/3)x - 64/3$ or $y = (8/3)x + 48/3 - 64/3$ or $y = (8/3)x - 16/3$, so $a = 8/3$ and $b = -16/3$.

3. (a) We first try evaluating the expression at $x = 8$. We obtain

$$\frac{\sqrt{5(8)-4}-1}{3(8^2)+2} = \frac{\sqrt{36}-1}{3(64)+2} = \frac{5}{194}.$$

   Since evaluating the expression gives a number, not a fraction with zero in the denominator, we see that we could apply the limit laws to obtain the same result (try it if you feel up to it; see the notes and the textbook for help). In any case, the final answer is $5/194$.

   (b) In this case if we try to substitute the value $x = 0$ into the expression under the limit we get $0/0$ which doesn't make sense, so we know we have to do some preprocessing before we can apply limit laws to evaluate the limit. We write

$$\lim_{x\to 0}\frac{x^2+3x}{x} = \lim_{x\to 0}\frac{x(x+3)}{x} = \lim_{x\to 0} x+3 = \lim_{x\to 0} x + \lim_{x\to 0} 3 = 0+3 = 3$$

   where limit laws were employed in the second last step. (I don't mind if you don't apply the limit laws in detail; just substituting $x = 0$ into $x+3$ is fine, but you should at least understand that the limit laws were used.)

   (c) Again, if we substitute $x = 3$ into the expression under the limit we get $(3^2-3-6)/(3-3) = 0/0$ so we need to preprocess the expression. Factoring, we have

$$\lim_{x\to 3}\frac{x^2-x-6}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} x+2 = 5.$$

   I didn't apply the limit laws in detail because just substituting $x = 3$ into $x+2$ gives the same result.

4. (a) Recall that the definition of the derivative is

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

   which makes sense if you remember the definition of the slope of the tangent line as the limit of the slopes of secant lines. (Otherwise, just memorize that formula.) In order to evaluate the above expression we set

$a = 3$ and so we need to find

$$\begin{aligned} f(3) &= 3^2 + 2 = 11 \\ f(3+h) &= (3+h)^2 + 2 = 9 + 6h + h^2 + 2 = 11 + 6h + h^2 \\ f(3+h) - f(3) &= 11 + 6h + h^2 - 11 = 6h + h^2 \\ \frac{f(3+h)-f(3)}{h} &= \frac{6h+h^2}{h} = \frac{h(6+h)}{h} = 6 + h. \end{aligned}$$

Now with all the work we did above, finding the limit from the definition is easy:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} 6 + h = 6.$$

You should check that the answer agrees with what you would obtain by other methods.

(b) The idea here is the same as for 4(b), but the algebra is just a little harder. We have

$$\begin{aligned} f(4) &= \frac{1}{2(4)+3} = \frac{1}{8+3} = \frac{1}{11} \\ f(4+h) &= \frac{1}{2(4+h)+3} = \frac{1}{11+2h} \\ f(4+h) - f(4) &= \frac{1}{11+2h} - \frac{1}{11} = \frac{11}{11(11+2h)} - \frac{11+2h}{11(11+2h)} = -\frac{2h}{11(11+2h)} \\ \frac{f(4+h)-f(4)}{h} &= \frac{-\frac{2h}{11(11+2h)}}{h} = -\frac{2h}{11(11+2h)} \cdot \frac{1}{h} = -\frac{2}{121+22h} \\ f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} -\frac{2}{121+22h} = -\frac{2}{121+22(0)} = -\frac{2}{121}. \end{aligned}$$

You should check that the above answer agrees with what you would obtain using the general power rule.

(c) As above, we have

$$\begin{aligned} f(2) &= \sqrt{5-2} = \sqrt{3} \\ f(2+h) &= \sqrt{5-(2+h)} = \sqrt{3-h} \\ f(2+h) - f(2) &= \sqrt{3-h} - \sqrt{3} \\ f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{3-h}-\sqrt{3}}{h}. \end{aligned}$$

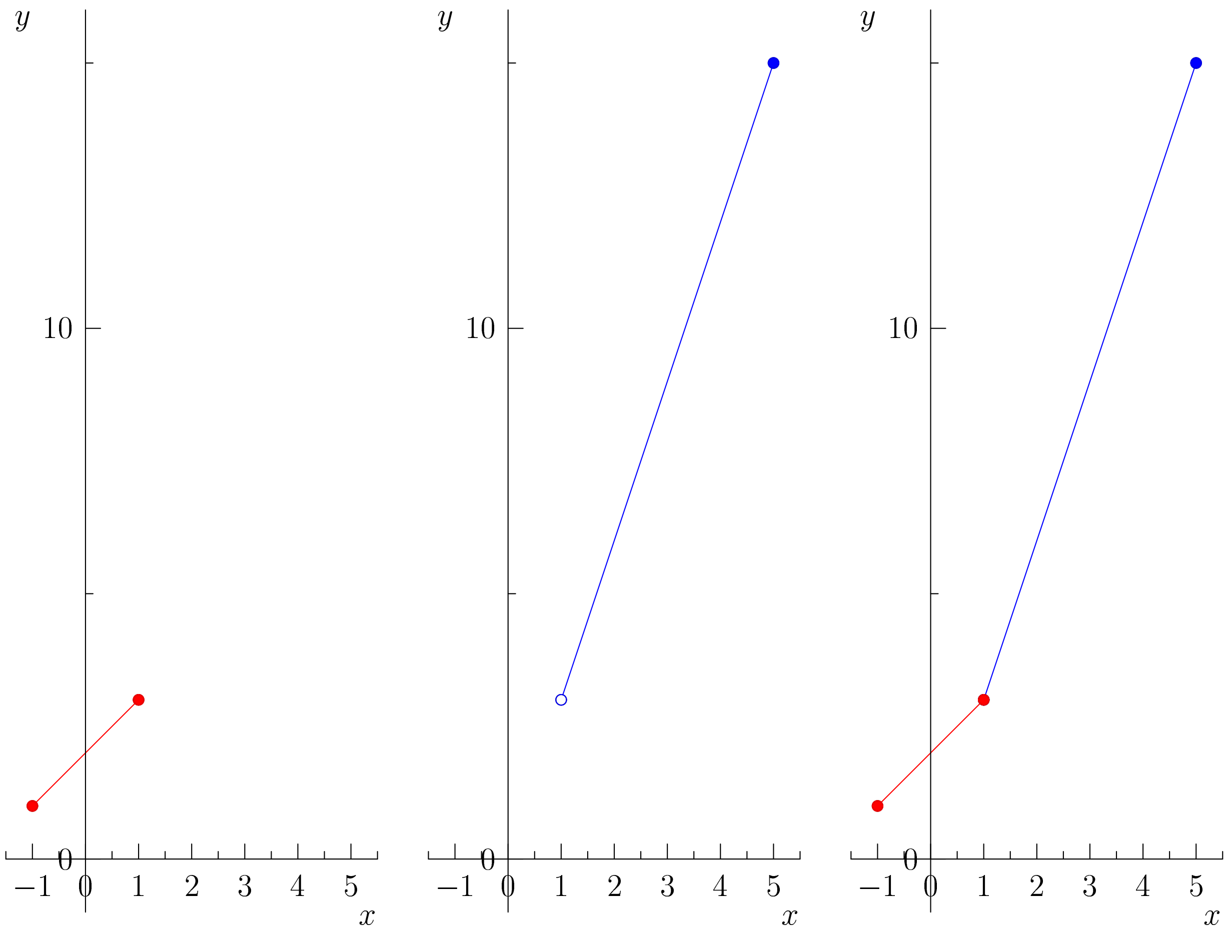
In order to evaluate the above limit we need to use the trick of multiplying by the conjugate radical:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{\sqrt{3-h}-\sqrt{3}}{h} = \lim_{h\to 0} \frac{(\sqrt{3-h}-\sqrt{3})(\sqrt{3-h}+\sqrt{3})}{h(\sqrt{3-h}+\sqrt{3})} = \lim_{h\to 0} \frac{(3-h)-3}{h(\sqrt{3-h}+\sqrt{3})} \\ &= \lim_{h\to 0} \frac{-h}{h(\sqrt{3-h}+\sqrt{3})} = \lim_{h\to 0} \frac{-1}{\sqrt{3-h}+\sqrt{3}} = \frac{-1}{\sqrt{3}+\sqrt{3}} = \frac{-1}{2\sqrt{3}}. \end{aligned}$$

You should check that the answer agrees with what you would obtain using the general power rule.

5. It is best to first graph the functions, and then try to figure out from the graphs whether they are continuous and/or differentiable.

(a) We can graph f by graphing each of the cases and then combining the graphs together; see Figure 1(i) for the case $-1 \le x \le 1$, Figure 1(ii) for the case $1 < x \le 5$, and Figure 1(iii) for the combination of the two. From the latter graph, we can see that the function is continuous (because the graph can be drawn without lifting the pen from the page), but that it is not differentiable at $x = 1$ because there is a sharp change in direction in the graph at that point.



(i) $y = x + 2$ for $-1 \leq x \leq 1$ (ii) $y = 3x$ for $1 < x \leq 5$ (iii) $y = f(x)$ for $-1 \leq x \leq 5$

Figure 1: Graphs for question 5(a)

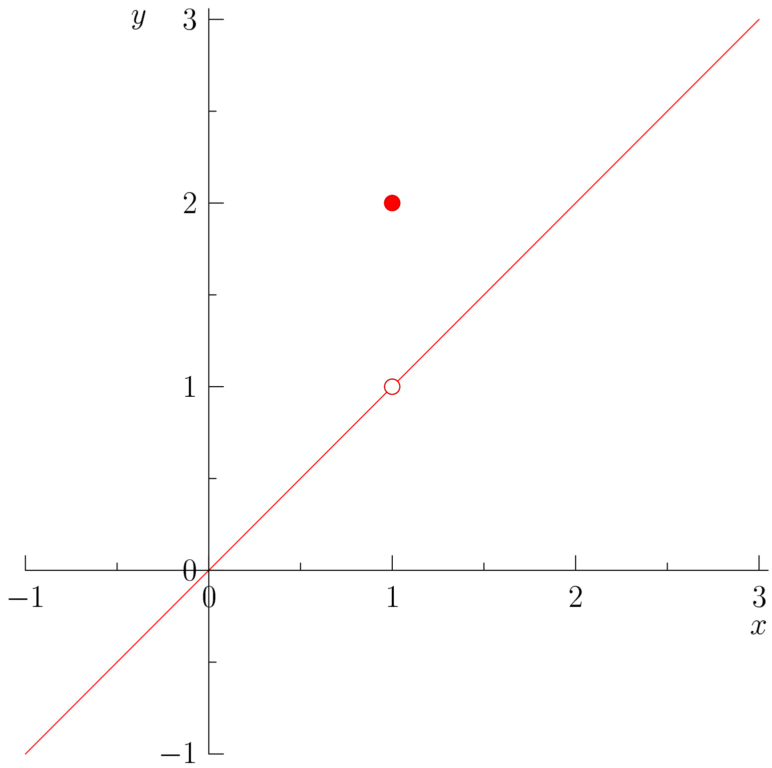


Figure 2: Graph for question 5(b)

(b) In this case we graph the function $f(x) = x$ leaving a hole at the point $(1,1)$, then we put a dot at the point $(1,2)$ which is the special case given in the definition of the function. See Figure 2. From the figure, we see that $f(x)$ is not continuous at $x = 1$ because we have to lift the pen off the page to draw the function near that point. Furthermore, since the function is not continuous at that point, it can't be differentiable there either. You can also see that by noting that secant lines through the points $(1, f(1)) = (1,2)$ and $(1+h, f(1+h)) = (1+h, 1+h)$ become progressively steeper and steeper until they are nearly vertical as h tends to 0, so the derivative as the limit of the slopes of the secant lines can't exist there.

6. The idea here is that f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$. Therefore if f isn't defined at a, the only way to extend the definition of f to a which makes the function continuous is to define $f(a)$ to be the above limit.

(a) We have $f(x)$ undefined at $x = -4$, and in fact the formula provided doesn't make sense for $x = -4$, so it is not immediately obvious how to extend the definition of $f(x)$ to include $x = -4$. This is where limits are useful. We evaluate

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{x^2 + x - 12}{x+4} = \lim_{x\to -4} \frac{(x+4)(x-3)}{x+4} = \lim_{x\to -4} (x-3) = -7$$

so if we extend the definition of the function f to include $f(-4) = -7$, the resulting new function is continuous. (Strictly speaking, the new function should be named with a different letter, g say, because it is a different function from f. The definition of g is

$$g(x) = \begin{cases} \frac{x^2+x-12}{x+4}, & x \neq -4 \\ -7, & x = -4 \end{cases}$$

which is different from the definition of f. The function g could also be defined more simply as $g(x) = x - 3$; why?

(b) Again, we evaluate the limit as x approaches 5 to obtain

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{x^2+25}{x-5}.$$

Unfortunately, the numerator tends to a non-zero value but the denominator tends to 0, so the limit doesn't exist. (Try putting in some values close to 5 with your calculator; you'll find out that the resulting values of $f(x)$ grow larger and larger as we get close to 5; we say that the limit doesn't exist, and in particular that the limit is $\pm\infty$, but don't worry about that too much just now.) In any case, the function can't be patched up to be continuous, no matter how we define $f(5)$.

(c) As above, we need to evaluate the limit

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{9+x}-\sqrt{9}}{x}.$$

We evaluate the above limit by the trick of multiplying both the numerator and denominator of the above fraction by the conjugate radical $\sqrt{9+x}+\sqrt{9}$:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(\sqrt{9+x}-\sqrt{9})(\sqrt{9+x}+\sqrt{9})}{x(\sqrt{9+x}+\sqrt{9})} = \lim_{x\to 0} \frac{(9+x)-9}{x(\sqrt{9+x}+\sqrt{9})} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+\sqrt{9})} = \lim_{x\to 0} \frac{1}{\sqrt{9+x}+\sqrt{9}}.$$

Now we can substitute the value $x=0$ into the above limit to obtain

$$\lim_{x\to 0} f(x) = \frac{1}{\sqrt{9+0}+\sqrt{9}} = \frac{1}{\sqrt{9}+\sqrt{9}} = \frac{1}{6}.$$

So extending the definition of f to include $f(0)=1/6$ results in a continuous function.

7. (a) See Figure 3. Note that both curves pass through the point $(1,1)$. The tangent lines for each of the curves is dashed in the appropriate colour.

(b) We have $f'(x)=2x$ so $f'(1)=2$, and $g'(x)=3x^2$ so $g'(1)=3$. This means that f is increasing more slowly than g at the point $(1,1)$, i.e., where the two graphs cross. In geometric terms, the graph of g crosses the graph of f from below to above at $(1,1)$.

8. (a) We can build up the function R by looking at two cases: the number of copies is less than or equal to 150, and the number of copies is greater than 150. Then we have

$$R(x) = \begin{cases} 2.50+0.10x & \text{for } x \le 150 \\ 2.50+0.10(150)+0.07(x-150) & \text{for } x > 150 \end{cases} = \begin{cases} 2.50+0.10x & \text{for } x \le 150 \\ 7.00+0.07x & \text{for } x > 150 \end{cases}$$

where the latter expression was found by simplifying the formula for the second case with simple algebra.

(b) The profit is revenue minus cost, or $P(x)=R(x)-C(x)$. Since $C(x)$ has the simple formula $C(x)=0.03x$ we can write

$$R(x) = \begin{cases} 2.50+0.10x & \text{for } x \le 150 \\ 7.00+0.07x & \text{for } x > 150 \end{cases} - C(x) = \begin{cases} 2.50+0.10x-0.03x & \text{for } x \le 150 \\ 7.00+0.07x-0.03x & \text{for } x > 150 \end{cases} = \begin{cases} 2.50+0.07x & \text{for } x \le 150 \\ 7.00+0.04x & \text{for } x > 150 \end{cases}$$

(c) The profit function is continuous but not differentiable; in particular, it fails to be differentiable at $x=150$. See Figure 4.

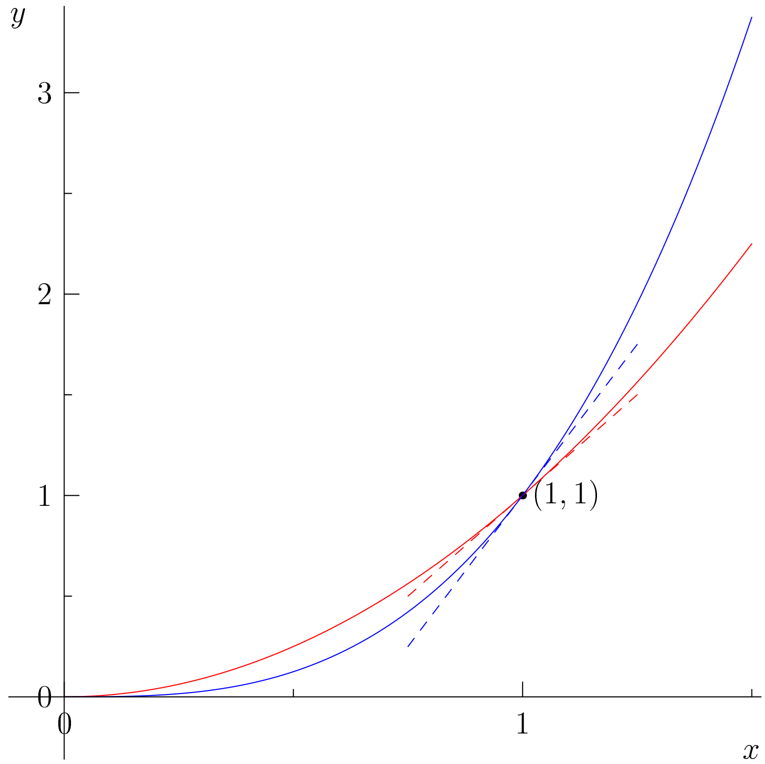


Figure 3: Graph for question 7a

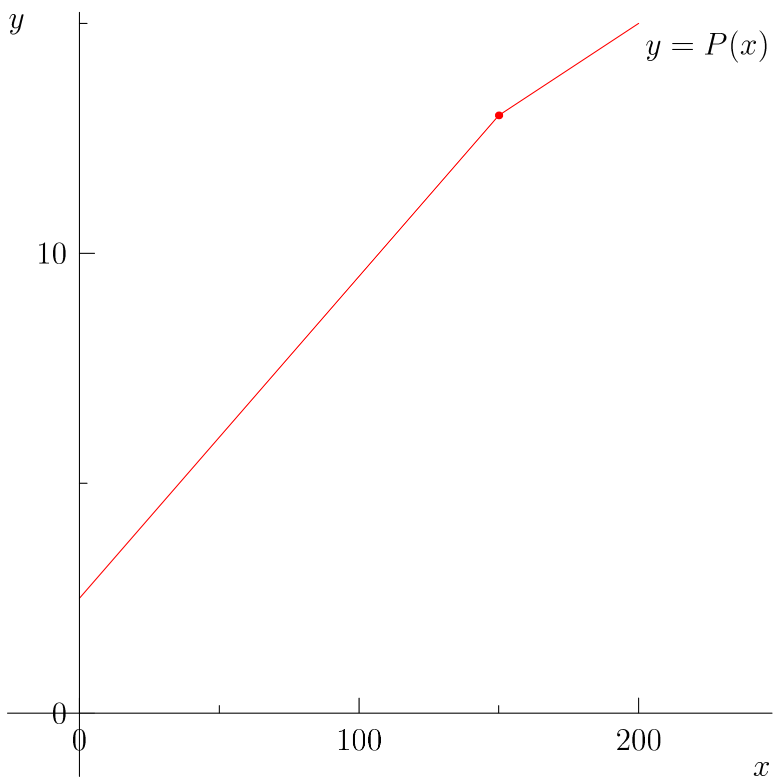


Figure 4: Graph for question 8(c)