MATH 103 Problem Set 4 Solutions DRAFT

Edward Doolittle

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1. (a) The first derivative is $f'(x) = 3x^2 - 27$. Stationary points are where $f'(x) = 0$, i.e., where $3x^2 - 27 = 0$. Solving, $x = \pm 3$ are the stationary points.

   (b) The first derivative is $g'(x) = -18x^2 - 3x + 3$. Stationary points are where $g'(x) = 0$, i.e., where $-18x^2 - 3x + 3 = 0$. Dividing through by the common factor of -3 we have $6x^2 + x - 1 = 0$. Factoring, $(3x - 1)(2x + 1) = 0$, so the critical points are $x = 1/3$ and $x = -1/2$.

   (c) The first derivative is $h'(x) = 6x^2 + 6x$, so the stationary points are where $6x^2 + 6x = 0$, i.e., where $6x(x + 1) = 0$, i.e., $x = 0$ and $x = -1$.

2. To apply the first derivative test, we need to check the sign of the first derivative at points near the stationary point. By near, I mean closer than any other stationary point (or any other "interesting" point, for that matter; for the moment, only stationary points are "interesting").

   (a) The stationary points are at $x = -3$ and $x = 3$, so we can check the first derivative at $x = -4$, $x = 0$, and $x = 4$. At $x = -4$, $f'(-4) = 3(-4)^2 - 27 = 21 > 0$ so the function is increasing at $x = -4$. At $x = 0$, $f'(0) = 3(0)^2 - 27 = -27 < 0$ so the function is decreasing at $x = 0$. At $x = 4$, $f'(4) = 3(4)^2 - 27 = 21 > 0$ so the function is increasing at $x = 4$. The function increases to $x = -3$, then decreases to $x = 3$, then increases to the right of $x = 3$. Therefore the function has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

   (b) The stationary points are at $x = -1/2$ and $x = 1/3$. We can test the first derivative at $x = -1$, $x = 0$, and $x = 1$. At $x = -1$, $g'(-1) = -18(-1)^2 - 3(-1) + 3 = -18 + 3 + 3 = -12 < 0$ so the function is decreasing. At $x = 0$, $g'(0) = -18(0)^2 - 3(0) + 3 = 3 > 0$ so the function is increasing. At $x = 1$, $g'(1) = -18(1)^2 - 3(1) + 3 = -18 < 0$ so the function is decreasing. The function decreases to $x = -1/2$, increases from $-1/2$ to $1/3$, and then decreases to the right of $1/3$. Therefore it has a local minimum at $x = -1/2$ and a local maximum at $x = 1/3$.

   (c) The stationary points are at $x = -1$ and $x = 0$. We can test the first derivative at $x = -2$, $x = -1/2$, and $x = 1$. We have $h'(-2) = 6(-2)^2 + 6(-2) = 12 > 0$ so the function is increasing to the left of $x = -1$; we have $h'(-1/2) = 6(-1/2)^2 + 6(-1/2) = 3/2 - 6/2 = -3/2 < 0$ so the function is decreasing between $x = -1$ and $x = 0$; and we have $h'(1) = 6(1)^2 + 6(1) = 12 > 0$ so the function is increasing to the right of $x = 0$.

3. (a) The second derivative is $f''(x) = 6x$. The inflection points are where $f''(x) = 0$, i.e., where $6x = 0$, i.e., at $x = 0$. At the stationary point $x = -3$, $f''(-3) = 6(-3) = -18 < 0$, so the function is concave down at $x = -3$, which confirms that the function has a local maximum at that stationary point by the second derivative test. At the stationary point $x = 3$, $f''(3) = 6(3) = 18 > 0$, so the function is concave up at $x = 3$, which confirms that the function has a local minimum at that stationary point.

   (b) The second derivative is $g''(x) = -36x - 3$. The inflection points are where $g''(x) = 0$, i.e., where $-36x - 3 = 0$, i.e., $x = -1/12$. Furthermore, at the stationary point $x = -1/2$ we have $g''(-1/2) = 18 - 3 = 15 > 0$, concave up, local minimum, and at the stationary point $x = 1/3$ we have $g''(1/3) = -12 - 3 = -15 < 0$, concave down, local maximum.

   (c) The second derivative is $h''(x) = 12x + 6$ so the inflection point is where $12x + 6 = 0$, i.e., $x = -1/2$. At the stationary points we have $h''(-1) = -6 < 0$, concave down, local maximum and $h''(1) = 6 > 0$, concave up, local minimum.

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5. The x-intercepts are the locations where the graph of the function crosses the x-axis.

   (a) To find the x-intercepts, we have to solve the equation $R(x) = 0$, i.e.,

   $$\frac{12}{x} + 3x + 1 = 0 \implies 3x^2 + x + 12 = 0.$$

   By the quadratic formula

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(3)(12)}}{2(3)},$$

   we see that the quadratic equation has no roots (we can't take the square root of a negative number) and hence the graph has no x-intercepts.

   (b) Clearing fractions as in the previous, we need to solve

   $$\frac{1}{x^2} + \frac{x}{4} - \frac{5}{4} = 0 \implies 4 + x^3 - 5x^2 = 0.$$

   Normally we would have great difficulty solving a cubic equation like $x^3 - 5x^2 + 4 = 0$, but we are given a hint that $x = 1$ is a root (which we can check), so we can factor out $x - 1$ to obtain $(x - 1)(x^2 - 4x - 4) = 0$, and we've reduced the problem to that of factoring the quadratic $x^2 - 4x - 4$. Using the quadratic formula, the roots are

   $$x = \frac{4 \pm \sqrt{16 - 4(1)(-4)}}{2} = 2 \pm 2\sqrt{2},$$

   approximately $x = 4.83$ and $x = -0.83$.

   (c) As usual, we need to solve the equation $P(x) = 0$, i.e.,

   $$\frac{1}{\sqrt{x}} + \frac{x}{2} = 0.$$

   However, both terms are positive for $x > 0$ so there can be no solution, i.e., the curve does not intercept the x-axis.

6. (a) The stationary points are where $R'(x) = 0$, i.e., $-12/x^2 + 3 = 0$, i.e., $x^2 = 4$, i.e., $x = 2$. (We throw away $x = -2$ because we are given $x > 0$ as a restriction on the domain of the function.) The corresponding y value is $y = R(2) = 12/2 + 3(2) + 1 = 13$, so a stationary point is located on the graph at $(2, 13)$.

   The extreme points are the stationary points at which the function passes the first or second derivative test. The second derivative test is easier to apply in this case: $R''(x) = 24/x^3$, $R''(2) > 0$ so $(2, 13)$ is in fact a local minimum point.

   The inflection points are where $R''(x) = 0$, i.e., $24/x^3 = 0$. There is no inflection point because that equation has no solution. Many students make the mistake of saying that $24/x^3 = 0$ implies $x = 0$, but that is not true; substituting $x = 0$ into the expression $24/x^3$ gives an undefined result, not 0.

   (b) The derivatives of C are $C'(x) = -2x^{-3} + 1/4$ and $C''(x) = 6x^{-4}$. Stationary points are where $C'(x) = 0$, i.e., $-2x^{-3} + 1/4 = 0$, i.e., $x^3 = 8$, i.e. $x = 2$. The stationary point is an extreme point, a minimum in fact, by the second derivative test: $C''(2) > 0$. The location of the minimum on the graph is $(2, C(2)) = (2, -1/2)$. Finally, inflection points are where $C''(x) = 6x^{-4} = 0$, i.e., nowhere.

   (c) We have $P'(x) = (-1/2)x^{-3/2} + (1/2)$ and $P''(x) = (3/4)x^{-5/2}$. For stationary points $P'(x) = 0$ which implies $x^{-3/2} = 1$ which implies $x = 1^{-2/3} = 1$. The stationary point is a minimum by the second derivative test because $P''(1) > 0$. The stationary point on the graph is located at $(1, P(1)) = (1, 3/2)$. There is no inflection point because $P''(x) = (3/4)x^{-5/2} = 0$ has no solutions.

7. In each case, it is actually easy to find the asymptotes: we look for a term which ‘misbehaves’, and that helps us find the asymptotes. In each case it is the first term which misbehaves.

    (a) In this case, the first term $12/x$ blows up as x approaches 0, so there is a vertical asymptote at $x = 0$. Furthermore, the term $12/x$ is negligible for large x, so $R(x) \approx 3x + 1$ for large x, i.e., $y = 3x + 1$ is a slant asymptote.

    (b) The first term $1/x^2$ blows up as x approaches 0, so there is a vertical asymptote at $x = 0$. In addition, the term $1/x^2$ is negligible for large x, so $y = x/4 - 5/4$ is a slant asymptote.

    (c) The first term $1/\sqrt{x}$ blows up as x approaches 0, so there is a vertical asymptote at $x = 0$. Furthermore, the term $1/\sqrt{x}$ is negligible for large x, so $y = x/2$ is a slant asymptote.

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