MATH 103 Problem Set 5 Solutions DRAFT

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1. (a) Differentiating, $f'(t) = 24 - 2t$. Setting the first derivative equal to zero, $24 - 2t = 0$ implies $t = 12$. By the second derivative test, $f''(t) = -2 < 0$ when $t = 12$, so f has a (local) maximum at $t = 12$. A quick sketch of the function, or application of the first derivative test, should convince you that the local maximum is actually a global maximum. The value of f at the local maximum is $f(12) = 24(12) - (12)^2 = 144$. In conclusion, the maximum value of f is 144, and that maximum occurs when $t = 12$.

   (b) This is similar to part (a): $g'(x) = 4 - 2x$ is zero when $x = 2$; $g''(x) = -2 < 0$ when $x = 2$ so the stationary point is a local maximum. A graph or the first derivative test should convince you that the local maximum really is a global maximum. In conclusion, the maximum value of the function is $g(2) = 10 + 4(2) - 2^2 = 14$ which occurs when $x = 2$.

   (c) This question is a little harder than the previous. Taking the derivative is still straightforward: $h'(t) = -3t^2 + 12t$. To find the stationary points we need to solve the equation $h'(t) = 0$ which implies $-3t^2 + 12t = 0$. Factoring the quadratic, we have $-3t(t-4) = 0$ so the roots are $t = 0$, $t = 4$. Now we really do need to check which is a local maximum and which is a local minimum. The second derivative is $h''(t) = -6t + 12$; we have $h''(0) = 12 > 0$ so the stationary point at $t = 0$ is a local minimum; similarly we have $h''(4) = -6(4) + 12 = -12 < 0$ so the stationary point at $t = 4$ is a local maximum.

   In conclusion, one would suspect that the maximum value of the function is $h(4) = -4^3 + 6(4)^2 + 40 = 72$, occuring at $t = 4$. However, the situation is more subtle than that. It is best to draw a graph of the function. The function $-t^3 + 6t^2 + 40$ actually does not have a maximum, because for large negative values of t, the function increases without bound. However, the function $h(t)$ does have a maximum, because those large negative values are not allowed. That is one reason graphing is important: sometimes the global picture shows us things that the local picture doesn't.

2. (a) We use the constraint equation to solve for one of the unknown variables in terms of the other. In this case. x and y are interchangeable, so we can choose to write either one in terms of the other. Since we often use x as the independent unknown, let's write y in terms of x: $y = 2 - x$. We can now eliminate y from the objective function to obtain $Q = xy = x(2 - x)$. Optimizing $Q(x) = x(2-x) = 2x - x^2$, we have $Q'(x) = 2 - 2x$, $Q''(x) = -2$, from which we find that a local maximum for Q occurs at $x = 1$ (why?). At the local maximum we have $x = 1$, $y = 2 - x = 2 - 1 = 1$, and $Q = xy = 1 \cdot 1 = 1$.

   (b) Again, we use the constraint equation to eliminate one of the variables. In this case, the situation is not completely symmetric, so it may be that eliminating one of x or y is a better choice than eliminating the other. In fact, the calculations are slightly easier if we eliminate x by writing it in terms of y. (Try it the other way to compare.) By the constraint we have $x = 4 - y$ so $Q = xy^2 = (4 - y)y^2 = 4y^2 - y^3$. Differentiating, $Q'(y) = 8y - 3y^2$, $Q''(y) = 8 - 6y$; stationary points are when $Q'(y) = 0$ which implies $(8 - 3y)y = 0$ which implies $y = 0$ or $y = 8/3$. The second derivative test tells us that $Q(y)$ has a local minimum at $y = 0$ and a local maximum at $y = 8/3$. The first derivative test tells us that Q is decreasing from $y = 8/3$ to $y = 4$ (the upper limit on allowable values of y), so $y = 8/3$ gives a global maximum. At that value of y we have $x = 4 - (8/3) = 4/3$ and finally $Q = (4/3)(8/3)^2 = (4/3)(64/9) = 256/27$, the answer to the question. Draw a graph of $Q(y)$ to understand better what is happening.

   (c) The situation here is symmetric in x and y, so it doesn't matter which variable is eliminated. I'm partial to x (it uses less ink) so let's solve for y: $y = 6 - x$ according to the constraint, so $Q(x) = -x^2 - (6 - x)^2 = -x^2 - (36 - 12x + x^2) = -x^2 - 36 + 12x - x^2 = -2x^2 + 12x - 36$. Differentiating to the find the maximum, $Q'(x) = -4x + 12$, and while we're thinking of it, $Q''(x) = -4$. At the maximum we must have $Q'(x) = -4x+12 = 0$ which implies $4x = 12$ which implies $x = 3$. Since $Q''(3) < 0$ there is a local maximum at $x = 3$. The first derivative test or a graph shows that the local maximum really is a global maximum. At that value of x we have $y = 6-x = 6-3 = 3$ and $Q = -x^2-y^2 = -3^2-3^2 = -9-9 = -18$.

3. Call the length of the side of the garden that faces the road x, and call the length of a side of the garden that is perpendicular to the road y.

   (a) The objective function is the area $A = xy$; the constraint is that the cost $8x+6y+6x+6y$ must be 800 dollars, i.e., $14x+12y = 800$.

   (b) We first need to use the constraint to find the length of the side perpendicular to the road in terms of the length of the side parallel to the road; i.e., we have to solve for y in terms of x. We have $12y = 800-14x$ or $y = 200/3-(7/6)x$. Now we can express the objective in terms of x alone: $A = xy = x(200/3-(7/6)x) = (200/3)x-(7/6)x^2$.

   (c) Differentiating the area in terms of x, we have $A'(x) = (200/3)-(7/3)x$ and $A''(x) = -7/3$. We have a stationary point where $A'(x) = 0$ which implies $(200/3)-(7/3)x = 0$ which implies $x = 200/7$. Since $A''(200/7) = -7/3 < 0$, A has a local maximum at that stationary point. The dimension of the perpendicular side is then $y = 200/3-(7/6)(200/7) = 100/3$. Therefore the dimensions of the garden which maximize the area are x (the side parallel to the road) is $200/7$ metres, and y (the side perpendicular to the road) is $100/3$ metres.

4. In contrast to the previous problem, here the area is the constraint and the cost is the objective. In detail, let x be the length of the side containing the fence, and let y be the length of a side perpendicular to the fence. The objective is to minimize $C = 45x+60y$ subject to the constraint $xy = 300$. Using the constraint to solve for y we have $y = 300/x$ so $C(x) = 45x+60(300/x) = 45x+18000/x$. Differentiating, $C'(x) = 45-18000/x^2$. Solving $C'(x) = 0$ we have $45-18000/x^2 = 0$, $x^2 = 400$, $x = \pm 20$. A negative length doesn't make sense, so the only stationary point $x = 20$. We have $C''(x) = 36000/x^3$ so $C''(20) > 0$ and we have a local minimum. (A graph or the first derivative test should convince you that it's a global minimum, but isn't necessary to answer this question.) The corresponding y dimension is found from the constraint: $y = 300/20 = 15$. In summary, the dimensions of the garden which minimize the cost are $x = 20$ metres (the direction parallel to the fence) and $y = 15$ metres (the direction perpendicular to the fence).

5. (a) Let x be the size of an order, so that the number of orders per year is $4000/x$. Therefore the cost of processing the orders is $40(4000/x) = 160000/x$. The average number of books in stock is $x/2$ (exactly half the time the store has more than $x/2$ books in stock, and the other half of the time the store has fewer than $x/2$ books in stock), so the carrying cost is $2(x/2) = x$. Altogether, the total cost for ordering and carrying the books is $C(x) = 160000/x+x$. To find the minimum, we differentiate and obtain $C'(x) = -160000/x^2+1$. Solving $C'(x) = 0$, we have $-160000/x^2+1 = 0$ which implies $x^2 = 160000$ which implies $x = \pm 400$. The negative root doesn't make sense, so the only stationary point for this problem is $x = 400$. At that point we have $C''(400) = 320000/(400)^3 > 0$ so it is a local minimum. (You can check whether it's a global minimum if you like, but that's not necessary.) In summary, the economic order quantity is 400 books, and 10 orders per year are required.

   (b) If the demand quadruples, the cost becomes $C(x) = 640000/x+x$ (why?), and following the same analysis as above, the EOQ is 800, the number of orders is 20. Note that if the demand quadruples, it isn't optimal to quadruple the size of the order; the best thing to do is double the size of the orders and double the frequency of the orders.

6. I did this problem during the lecture on Tuesday.

7. Let x be the number of memberships sold in excess of 500. Then The revenue is $R(x) = (100-x)(500+x) = 50000-400x-x^2$. The revenue is stationary when $R'(x) = -400-2x = 0$ which implies $x = -200$ (a maximum—why?). A simple interpretation of that result is that we want to sell $500+(-200) = 300$ memberships at 300 dollars each.

   On the other hand, the question doesn't say exactly what happens if fewer than 500 people join; it is implied that perhaps the club won't go at all if fewer than 500 people join. So think of it this way: the marginal revenue is negative for $x \geq 0$, so revenue is maximized when $x \geq 0$ is as small as possible. (You could say that for every membership sold over 500, you gain about \$100 in revenue but lose about $500 \times \$1$ on the discount; calculus makes that kind of reasoning precise.)

So, with the given information, it seems that the club would want to sell exactly 500 memberships, and we ought to investigate further what happens if fewer than 500 memberships are sold. Would people really be willing to pay \$300 for a membership, or is the demand curve not so simple and linear?

8. (a) To find the linear demand function, we can write it in point-slope form. Since $p = 8$ when $x = 1200$, we have $p - 8 = m(x - 1200)$. Now we need to find the slope m of the demand function. We do that by finding the rise over the run:

$$m = \frac{\text{rise}}{\text{run}} = \frac{8 - 7}{1200 - 1700} = -\frac{1}{500}$$

Note that the slope is negative (the more admissions you want to sell the lower the price has to be); remember to put everything in the right order in the slope formula, so the 1200 should be below the 8 not the 7. Also note that we are thinking of the price as the "y"-variable on the graph, so the prices should go in the numerator.

Anyway, we have a formula for the slope:

$$p - 8 = -\frac{1}{500}(x - 1200).$$

We should solve for p as a function of x to obtain

$$p = 8 - \frac{1}{500}(x - 1200).$$

It may or may not help to simplify. It is an easy task in this case and may help us later, so I suggest you do:

$$p = 8 - \frac{1}{500}x + \frac{1200}{500} = 10.4 - \frac{1}{500}x.$$

In summary, our demand function is $p = 10.4 - 0.002x$. You should check that that demand function is consistent with the data we have.

(b) The revenue function is $R(x) = xp = x(10.4 - 0.002x)$. Simplifying, we have $R(x) = 10.4x - 0.002x^2$. Revenue is maximized when $R'(x) = 10.4 - 0.004x = 0$ which implies $(4/1000)x = 10.4$ which implies $x = 2600$. The second derivative test $R''(2600) = -0.004 < 0$ tells us that we have a local maximum when $x = 2600$. (From the first derivative test or a graph you should be able to conclude that the local maximum is in fact a global maximum.)

However, remember that the real question is what should the club charge. For that, we put the optimizing value of x into the demand function to obtain $p = 10.4 - 0.002(2600) = 5.20$, so the club should charge \$5.20 to maximize its profit. (It may be that they don't want to deal with fractions of a dollar; in that case, would \$5 or \$6 give better revenue?)

(c) The cost function is $C(x) = 2000 + 0.4x$. The profit function is

$$P(x) = R(x) - C(x) = 10.4x - 0.002x^2 - (2000 + 0.4x) = 10.4x - 0.002x^2 - 2000 - 0.4x = -2000 + 10x - 0.002x^2.$$

Profit is maximized when $P'(x) = 10 - 0.004x = 0$ which implies $x = 2500$. The nightclub should charge $p(2500) = 10.4 - 0.002(2500) = 5.40$ to maximize its profit.

(d) If fire regulations cap the number of customers allowed in the building to 2400, we need to look at the sign of the marginal profit to make a decision. From $x = 0$ to $x = 2400$, the marginal profit $P'(x) = 10 - 0.004x$ is positive, which means that adding more customers increases the profit. So we should add as many customers as we can without exceeding fire regulations (and the stationary point); i.e., we should aim for $x = 2400$ customers. To get that number of customers we should charge $p(2400) = 10.4 - 0.002(2400) = 5.60$.

9. Let x and y be the dimensions of the bottom of the package, and z its height, in centimetres. The constraint provided by the airline is $x+y+z=120$. The package is square-bottomed, providing the constraint $x=y$. Substituting that into the previous constraint gives $2x+z=120$. The objective is the volume which is $V=xyz$; solving for y and z in terms of x gives $V(x)=xx(120-2x)=120x^2-2x^3$. Stationary points are where $V'(x)=240x-6x^2=0$ which implies $x=0$ or $x=40$. We have $V''(x)=240-12x$ so $V''(0)>0$ giving a local min and $V''(40)<0$ giving a local max. (You can graph the function to convince yourself that that is a global maximum under the natural restriction $x>0$.) The corresponding value of z is $z=120-2(40)=40$. So the box of greatest volume satisfying all the constraints is a cube 40 cm on a side.