MATH 103 Problem Set 6 Solutions DRAFT

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1. (a) By the product rule,

$$y' = \left(\frac{d}{dx}(x^2+3)\right)(2x-1) + (x^2+3)\frac{d}{dx}(2x-1) = 2x(2x-1) + (x^2+3)(2).$$

There is no need to simplify, but if you do, the answer is $y' = 6x^2 - 2x + 6$. You can check your answer by multiplying out the product and then differentiating, if you like.

(b) By the product rule,

$$y' = 2x(7x-1)^2 + x^2\frac{d}{dx}(7x-1)^2.$$

The remaining derivative can be evaluated by the generalized power rule to give

$$y' = 2x(7x-1)^2 + x^2 \cdot 2(7x-1) \cdot \frac{d}{dx}(7x-1) = 2x(7x-1)^2 + x^2 \cdot 2(7x-1) \cdot 7.$$

Again, you can check by multiplying y out before taking the derivative if you like.

(c) By the generalized power rule,

$$\frac{d}{dx}[(x-1)(7x+2)]^4 = 4[(x-1)(7x+2)]^3 \frac{d}{dx}[(x-1)(7x+2)].$$

Now applying the product rule we have

$$\frac{d}{dx}[(x-1)(7x+2)]^4 = 4[(x-1)(7x+2)]^3[(7x+2) + (x-1)\cdot 7].$$

Checking by doing multiplying the original function out isn't really feasible in this case, but you can try if you like.

(d) First, we apply the product rule to obtain

$$\frac{d}{dx}x^7(3x^4+12x-1)^2 = 7x^6(3x^4+12x-1)^2 + x^7\frac{d}{dx}(3x^4+12x-1)^2.$$

Now we apply the general power rule to the remaining derivative to obtain

$$\frac{d}{dx}x^7(3x^4+12x-1)^2 = 7x^6(3x^4+12x-1)^2 + x^7 \cdot 2(3x^4+12x-1)\frac{d}{dx}(3x^4+12x-1).$$

Finally, differentiating the polynomial we have

$$\frac{d}{dx}x^7(3x^4+12x-1)^2 = 7x^6(3x^4+12x-1)^2 + x^7 \cdot 2(3x^4+12x-1)(12x^3+12).$$

Again, it is not really feasible to check the answer by multiplying everything out, but you can try if you like.

2. (a) By the quotient rule,

$$y' = \frac{(x^2+2x-2)\frac{d}{dx}(x^2+2x-1) - (x^2+2x-1)\frac{d}{dx}(x^2+2x-2)}{(x^2+2x-2)^2} = \frac{(x^2+2x-2)(2x+2) - (x^2+2x-1)(2x+2)}{(x^2+2x-2)^2}.$$

If we were using the derivative for another purpose, such as graphing y as a function of x, we should simplify the numerator of the above expression. However, since we just needed to find the derivative, we should probably leave it alone at this point.

(b) By the sum rule we can differentiate each summand separately, so we apply the quotient rule twice, once to each term:

$$y' = \frac{(x+1)\frac{d}{dx}1 - 1\frac{d}{dx}(x+1)}{(x+1)^2} + \frac{(x-1)\frac{d}{dx}1 - 1\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{-1}{(x+1)^2} + \frac{-1}{(x-1)^2}.$$

In this problem we can check our answer by rewriting the function as $y = (x+1)^{-1} + (x-1)^{-1}$ and using the general product rule.

(c) By the quotient rule the derivative is

$$y' = \frac{(x^2+1)^2 \cdot 2x - x^2 \cdot 2(x^2+1)(2x)}{(x^2+1)^4}.$$

Simplification is possible but isn't necessary for this question. You can check your answer by writing $y = x^2(x^2+1)^{-2}$ and using the product rule and the general power rule.

(d) By the quotient rule the derivative is

$$y' = \frac{(x+3)(x+4) - (x+2)\frac{d}{dx}[(x+3)(x+4)]}{(x+3)^2(x+4)^2} = \frac{(x+3)(x+4) - (x+2)[(x+4)+(x+3)]}{(x+3)^2(x+4)^2}$$

where the product rule was used in the last step. Again, you can check the result by writing $y = (x+2)[(x+3)(x+4)]^{-1}$ and using the product rule and the general power rule, or (better) by using logarithmic differentiation if you know that process.

3. (a) By the chain rule,

$$y' = 10(x^4+x^2)^9 \cdot \frac{d}{dx}(x^4+x^2) = 10(x^4+x^2)^9 \cdot (4x^3+2x).$$

(b) First, write $y = (x^2+1)^{1/2}$. Then by the chain rule,

$$y' = \frac{1}{2}(x^2+1)^{-1/2} \cdot \frac{d}{dx}(x^2+1) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x.$$

Simplification is possible but isn't necessary.

(c) We could use the quotient rule here, but it is probably faster to write $y = (1-x^2)^{-1}$ and use the chain rule:

$$y' = -(1-x^2)^{-2} \cdot \frac{d}{dx}(1-x^2) = -(1-x^2)^{-2} \cdot (-2x).$$

You can check by comparing with what you would have obtained by the quotient rule.

(d) First we apply the product rule to obtain

$$y' = 3x^2(x^3-2)^4 + x^3 \cdot \frac{d}{dx}(x^3-2)^4.$$

Now we apply the general power rule to obtain

$$y' = 3x^2(x^3-2)^4 + x^3 \cdot 4(x^3-2)^3 \cdot \frac{d}{dx}(x^3-2) = 3x^2(x^3-2)^4 + x^3 \cdot 4(x^3-2)^3 \cdot 3x^2.$$

4. By the quotient rule, the derivative is

$$y' = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}.$$

At the point $(2,3)$ we have $x = 2$ so $y' = -2/(2-1)^2 = -2$. The point-slope form of the tangent line is then $y-3 = -2(x-2)$. The equation of the tangent line can be simplified to $y = -2x+7$.

5. The function can be written as

$$y = \frac{x}{(2-x^2)^{1/2}}$$

so the derivative is

$$y' = \frac{(2-x^2)^{1/2}\cdot 1 - x\cdot\frac{1}{2}(2-x^2)^{-1/2}(-2x)}{(2-x^2)} = \frac{(2-x^2)^{1/2} + x^2(2-x^2)^{-1/2}}{2-x^2}.$$

When $x = 1$ we have

$$y' = \frac{(2-1)^{1/2} + 1^2(2-1)^{-1/2}}{2-1} = \frac{1+1}{1} = 2.$$

The point-slope form of the tangent line is $y-1 = 2(x-1)$.

6. To find the inflection points, we need to find the second derivative. By the quotient rule, the first derivative is

$$y' = \frac{(x^2+1)\frac{d}{dx}1 - 1\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}.$$

Applying the quotient rule a second time, the second derivative is

$$y'' = \frac{(x^2+1)^2\frac{d}{dx}(-2x) - (-2x)\frac{d}{dx}(x^2+1)^2}{(x^2+1)^4} = \frac{-2(x^2+1)^2 + 2x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}.$$

Simplifying by taking a common factor of x^2+1 from the numerator and denominator we have

$$y'' = \frac{-2(x^2+1)+8x^2}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3}.$$

Inflection points can be found by setting the second derivative equal to 0:

$$y'' = 0 \implies \frac{6x^2-2}{(x^2+1)^3} = 0 \implies 6x^2-2 = 0 \implies x^2 = \frac{1}{3} \implies x = \pm\frac{1}{\sqrt{3}}.$$

When $x = 1/\sqrt{3}$, we have $y = 1/((1/\sqrt{3})^2+1) = 1/(4/3) = 3/4$, and similarly when $x = -1/\sqrt{3}$, so the inflection points are $(1/\sqrt{3}, 3/4)$ and $(-1/\sqrt{3}, 3/4)$.

7. We re-write the function in power notation as

$$y = (x^2-6x+10)^{1/2}.$$

To find stationary points, we take the first derivative, by the chain rule:

$$y' = \frac{1}{2}(x^2-6x+10)^{-1/2}\frac{d}{dx}(x^2-6x+10) = \frac{1}{2}(x^2-6x+10)^{-1/2}(2x-6) = \frac{x-3}{\sqrt{x^2-6x+10}}.$$

Setting the first derivative equal to 0 we have

$$y' = 0 \implies \frac{x-3}{\sqrt{x^2-6x+10}} = 0 \implies x-3 = 0 \implies x = 3.$$

When $x = 3$ we have $y = \sqrt{3^2-6(3)+10} = \sqrt{1} = 1$, so the stationary point on the graph of the function is $(3,1)$.

8. By the quotient rule, the derivative is

$$y' = \frac{(x+1)^2 \cdot 4 - 4x \cdot 2(x+1)}{(x+1)^4} = \frac{4(x+1) - 8x}{(x+1)^3} = \frac{-4x+4}{(x+1)^3}.$$

By the quotient rule again we have

$$y'' = \frac{(x+1)^3 \cdot -4 - (-4x+4) \cdot 3(x+1)^2}{(x+1)^6} = \frac{-4(x+1) - (-12x+12)}{(x+1)^4} = \frac{8x-16}{(x+1)^4}.$$

The stationary points are where

$$y' = 0 \implies \frac{-4x+4}{(x+1)^3} = 0 \implies -4x+4 = 0 \implies x = 1.$$

The only stationary point is at $x = 1$, and $y''(1) = (8-16)(1+1)^4 < 0$ so the point is a local maximum. The location of the maximum on the graph is $(1, y(1)) = (1, 4/2^2) = (1, 1)$.

The inflection points are found by setting

$$y'' = 0 \implies \frac{8x-16}{(x+1)^4} = 0 \implies 8x - 16 = 0 \implies x = 2.$$

We have $y'' < 0$ to the left of the inflection point and $y'' > 0$ to the right of the inflection point.

Since y is undefined for $x = -1$, there is a likely a vertical asymptote $x = -1$. Since y becomes very small for large values of x (try it with a calculator if you don't believe me), there is a horizontal asymptote $y = 0$. See Figure 1 for the graph.

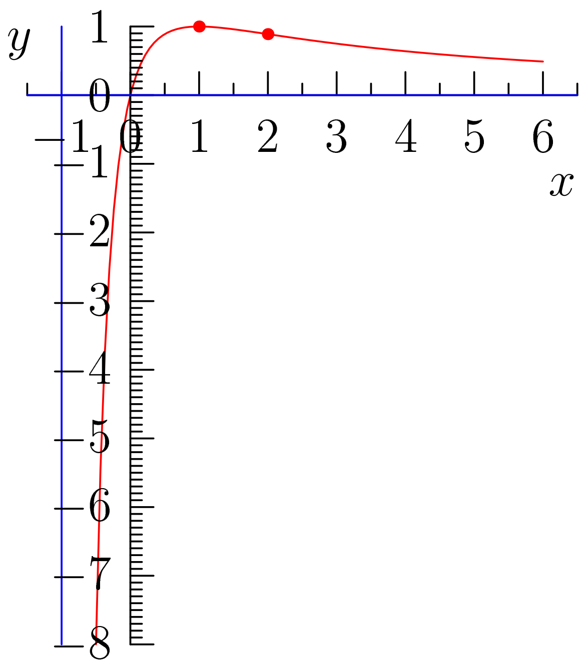

Figure 1: Graph of $4x/(x+1)^2$, $x > -1$

9. (a) The marginal profit per unit produced is

$$\frac{dP}{dx} = \frac{(100+x^2)(200) - (200x)(2x)}{(100+x^2)^2} = \frac{-200x^2 + 20000}{(100+x^2)^2}$$

where the numerator has been simplified. (Note that the stationary point for P as a function of x occurs when $x = 10$; it probably doesn't make sense to increas production beyond 10 units.)

(b) To find the time rate of change of the profit, we can do it in two different ways. The hard way is to express P as a function of time like this:

$$P = \frac{200x}{100+x^2} = \frac{200(4+2t)}{100+(4+2t)^2}$$

and then differentiate. The easy way is to use the chain rule

$$\frac{dP}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt}.$$

By part (a), we already know the first multiplicand in the above, and the second is easily calculated: $dx/dt = 2$. Therefore we can conclude that

$$\frac{dP}{dt} = \frac{-200x^2+20000}{(100+x^2)^2} \cdot 2.$$

Since we likely want to know dP/dt as a function of t, not x, we now substitute $x = 4+2t$ to obtain

$$\frac{dP}{dt} = 2\frac{-200(4+2t)^2+20000}{(100+(4+2t)^2)^2}.$$

It may be interesting to do the problem both ways and compare.

(c) The change in profit with respect to time when $t = 8$ is

$$\frac{dP}{dt} = 2\frac{-200(4+2(8))^2+20000}{(100+(4+2(8))^2)^2} = 2\frac{-200(20)^2+20000}{(100+20^2)^2} = 2\frac{-80000+20000}{500^2} = 2\frac{-60000}{250000} = -\frac{12}{25} = -0.48.$$

Profits are now actually declining; they probably should have stopped adding production after 3 weeks. (How did I get that number?)

10. To show that the total cost function is increasing, we find the derivative:

$$C'(x) = \frac{(x+100)(100x+10000)-(50x^2+10000x)}{(x+100)^2}.$$

We can conclude that $C(x)$ is increasing if we can show that $C'(x) > 0$. To do so, we should simplify $C'(x)$:

$$C'(x) = \frac{100x^2+20000x+1000000-50x^2-10000x}{(x+100)^2} = \frac{50x^2+20000x+1000000}{(x+100)^2}.$$

For $x > 0$, both the numerator and denominator of the above fraction are clearly positive, so $C'(x)$ is positive, so $C(x)$ is increasing. In conclusion, we see that as the level of production increases, the total cost of production increases.

To show that the average cost is decreasing, we first need an expression for the average cost, i.e. the total cost $C(x)$ divided by the number x of cases produced. We can write

$$A(x) = \frac{C(x)}{x} = \frac{5x(x+200)/(x+100)}{x}.$$

Using a little algebra to simplify, we have

$$A(x) = \frac{5(x+200)}{x+100} = \frac{5x+1000}{x+100}.$$

We can conclude that $A(x)$ is decreasing if we can show that $A'(x) < 0$. To do so, we calculate

$$A'(x) = \frac{(x+100)(5)-(5x+1000)(1)}{(x+100)^2} = \frac{5x+500-5x-1000}{(x+100)^2} = \frac{-500}{(x+100)^2}.$$

For any value of x, the numerator of the above expression is negative while the denominator is positive. Therefore $A'(x) < 0$ for all x, and so $A(x)$ is a decreasing function of x. I.e., the average cost per case of production decreases as the level of production increases.

It is interesting to get some estimates for the average cost of production. For small x, we have $x+200$ is approximately equal to 200, $x+100$ is approximately equal to 100, so $A(x)$ is approximately equal to $5(200)/100 = 10$ dollars per case. For large values of x, much larger than 200, we have $x+200$ is approximately x, $x+100$ is approximately x, so $A(x)$ is approximately $5x/x = 5$ dollars per case. You might want to graph $A(x)$ to verify those conclusions. It is also interesting to graph $C(x)$ and find the slant asymptote to the graph; what would you guess that the slant asymptote should be, given that $A(x) = C(x)/x$ is approximately equal to 5 for large x?