MATH 103 Problem Set 7 Solutions DRAFT

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1. (a) By the sum, constant multiple, and chain rules,

$$\frac{d}{dx}(1+4x+e^{-2x}) = \frac{d}{dx}1 + \frac{d}{dx}4x + \frac{d}{dx}e^{-2x} = 4 + e^{-2x}\frac{d}{dx}(-2x) = 4 - 2e^{-2x}.$$

(b) By the chain rule,

$$\frac{d}{dx}(e^{-3x}-2x)^4 = 4(e^{-3x}-2x)^3\frac{d}{dx}(e^{-3x}-2x) = 4(e^{-3x}-2x)^3(-3e^{-3x}-2),$$

where the chain rule was applied again in the last step.

(c) Here it helps if you first simplify the function using rules for exponentials, i.e.

$$e^t(e^{2t}-e^{-2t}) = e^te^{2t} - e^te^{-2t} = e^{t+2t} - e^{t-2t} = e^{3t} - e^{-t}.$$

Differentiating by the chain rule,

$$\frac{d}{dt}(e^{3t}-e^{-t}) = e^{3t}\frac{d}{dt}3t - e^{-t}\frac{d}{dt}(-t) = 3e^{3t} + e^{-t}.$$

(d) by the quotient rule,

$$\frac{d}{dx}\frac{4x^2}{x^2+e^{2x}} = \frac{(x^2+e^{2x})\cdot 8x - 4x^2\cdot(2x+2e^{2x})}{(x^2+e^{2x})^2}.$$

2. (a) Taking the natural logarithm of both sides,

$$\ln(e^{1-3x}) = \ln 4 \implies 1-3x = \ln 4 \implies 3x = \ln 4 - 1 \implies x = \frac{1}{3}(\ln 4 - 1).$$

(b) Exponentiating both sides,

$$e^{\ln x^2} = e^9 \implies x^2 = e^9 \implies x = (e^9)^{1/2} = e^{9/2}.$$

(c) Squaring both sides and then taking the natural logarithm,

$$(e^{\sqrt{x}})^2 = e^x \implies e^{2\sqrt{x}} = e^x \implies 2\sqrt{x} = x.$$

squaring again we get $4x = x^2$ which implies $x = 0$ or $x = 4$. Either of those solutions works, as can be seen by substituting into the original equation.

(d) First simplifying by laws for exponents, we have $(e^x)^2\cdot e^{2-3x} = e^{2x}\cdot e^{2-3x} = e^{2x+2-3x} = e^{2-x}$. Now the equation can be solved by taking logarithms:

$$\ln(e^{2-x}) = \ln 4 \implies 2-x = \ln 4 \implies x = 2 - \ln 4.$$

3. (a) Recall that the chain rule for logarithms can be summarized by

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

Applying that rule, we have

$$\frac{d}{dx}\ln(e^x - e^{-x}) = \frac{1}{e^x - e^{-x}}\frac{d}{dx}(e^x - e^{-x}) = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

(b) Applying the chain rule,

$$\frac{d}{dx}(1+\ln x)^3 = 3(1+\ln x)^2\frac{d}{dx}(1+\ln x) = 3(1+\ln x)^2\frac{1}{x}.$$

(c) By the quotient rule,

$$\frac{d}{dx}\frac{1}{\ln x} = \frac{\ln x \cdot \frac{d}{dx}1 - 1\frac{d}{dx}\ln x}{(\ln x)^2} = \frac{1/x}{(\ln x)^2} = \frac{1}{x(\ln x)^2}.$$

(d) By the chain rule,

$$\frac{d}{dx}\sqrt{\ln 2x} = \frac{d}{dx}(\ln 2x)^{1/2} = \frac{1}{2}(\ln 2x)^{-1/2}\frac{d}{dx}\ln 2x = \frac{1}{2}(\ln 2x)^{-1/2}\frac{1}{2x}\frac{d}{dx}2x = \frac{1}{2x}(\ln 2x)^{-1/2}.$$

4. (a) By the rule $\ln(xy) = \ln(x) + \ln(y)$ we have

$$\ln[(x+1)(2x-1)(4-3x)] = \ln(x+1) + \ln[(2x-1)(4-3x)] = \ln(x+1) + \ln(2x-1) + \ln(4-3x).$$

(It's OK to skip directly to the last step.) Now differentiating by the chain rule,

$$\frac{d}{dx}\ln[(x+1)(2x-1)(4-3x)] = \frac{1}{x+1}\frac{d}{dx}(x+1) + \frac{1}{2x-1}\frac{d}{dx}(2x-1) + \frac{1}{4-3x}\frac{3}{dx}(4-3x) = \frac{1}{x+1} + \frac{2}{2x-1} - \frac{3}{4-3x}.$$

(b) By reasoning similar to the previous question, and the rule $\ln(x^n) = n\ln(x)$, we have

$$\frac{d}{dx}\ln[(1+x)^3(2+x)(3+x)^2] = \frac{d}{dx}\left(3\ln(1+x) + \ln(2+x) + 2\ln(3+x)\right).$$

By the chain rule we have the above equal to

$$3\frac{1}{1+x}\frac{d}{dx}(1+x) + \frac{1}{2+x}\frac{d}{dx}(2+x) + 2\frac{1}{3+x}\frac{d}{dx}(3+x) = \frac{3}{1+x} + \frac{1}{2+x} + \frac{2}{3+x}.$$

(c) This is just a more complicated version of the previous. Recall the rule $\ln(x/y) = \ln(x) - \ln(y)$; then we have

$$\ln[x^5e^{4x}(3x+1)^{1/2}/(1-x^2)] = \ln(x^5) + \ln(e^{4x}) + \ln(3x+1)^{1/2} - \ln(1-x^2) = 5\ln x + 4x + \frac{1}{2}\ln(3x+1) - \ln(1-x^2).$$

Differentiating using the chain rule, we have the derivative of the above equal to

$$5\frac{1}{x} + 4 + \frac{1}{2}\frac{1}{3x+1}\frac{d}{dx}(3x+1) - \frac{1}{1-x^2}\frac{d}{dx}(1-x^2) = \frac{5}{x} + 4 + \frac{3}{2(3x+1)} + \frac{2x}{1-x^2}.$$

5. (a) Differentiating by the product rule, we have

$$\begin{aligned} y &= (1-x)e^{2x} \\ y' &= -e^{2x} + (1-x)2e^{2x} \\ y'' &= -2e^{2x} - 2e^{2x} + (1-x)4e^{2x}. \end{aligned}$$

Stationary points are where $y' = 0$ which implies

$$-e^{2x} + (1-x)2e^{2x} = 0 \implies (-1+2-2x)e^{2x} = 0. \implies (1-2x)e^{2x} = 0.$$

The above equation holds only if $1-2x = 0$ or $e^{2x} = 0$; the latter is impossible (the graph of e^{2x} never touches the x-axis), so we must have $x = 1/2$, which is the only stationary point. To determine the nature of the stationary point we plug that value into the second derivative to obtain

$$y''\left(\frac{1}{2}\right) = -2e^{2(1/2)} - 2e^{2(1/2)} + (1-(1/2))4e^{2(1/2)} = -4e + 2e = -2e < 0.$$

Since the second derivative is negative, the stationary point is a local maximum.

(b) We could use the quotient rule, but it's easier to write $y = (4x-1)e^{-x/2}$ and use the product rule. Then the answer to this question is similar to the previous.

(c) We have

$$\begin{aligned} y &= e^{-x} + 3x \\ y' &= -e^{-x} + 3 \\ y'' &= e^{-x}. \end{aligned}$$

Stationary points where $-e^{-x} + 3 = 0$ which implies $e^{-x} = 3$, $-x = \ln 3$, $x = -\ln 3$. Substituting that value into y'' we have $y''(-\ln 3) = e^{-(-\ln 3)} = e^{\ln 3} = 3 > 0$, so the stationary point is a local minimum.

(d) By the quotient rule we have

$$\begin{aligned} y &= \frac{x}{\ln x + x} \\ y' &= \frac{(\ln x + x) - x(1/x+1)}{(\ln x + x)^2} = \frac{\ln x + x - 1 - x}{(\ln x + x)^2} = \frac{\ln x - 1}{(\ln x + x)^2} \\ y'' &= \frac{(\ln x + x)^2(1/x) - (\ln x - 1)2(\ln x + x)(1/x+1)}{(\ln x + x)^3}. \end{aligned}$$

I took more care than usual to simplify the first derivative because the first derivative was used to calculate the second derivative, and will be used to find the stationary points. The stationary points are where $y' = 0$ which implies $(\ln x - 1)/(\ln x + x)^2 = 0$ which implies $\ln x - 1 = 0$ which implies $\ln x = 1$ which implies $x = e^{\ln x} = e^1 = e$. At that value of x we have $y''(e) = ((\ln e + e)^2(1/e) - (lne-1)X)/(\ln e + e)^3$, where I don't care what X is because I realized that $\ln e - 1 = 0$. Continuing evaluation of that expression we have $y''(e) = (1+e)^2(1/e)/(1+e)^3 > 0$, so the stationary point is a local minimum.

6. The rate of change of the value of the computer is

$$v'(t) = \frac{d}{dt}2000e^{-0.35t} = 2000e^{-0.35t}\frac{d}{dt}(-0.35t) = 2000(-0.35)e^{-0.35t} = -700e^{-0.35t}.$$

The rate of change of the value of the computer when $t = 3$ is $v'(3) = -700e^{-0.35\cdot 3} = -700e^{-1.05}$. My calculator tells me that is about -244.96, i.e., after 3 years the computer is losing value at a rate of about \$245 dollars per year.

7. (a) If we have $y = e^x(3x-4)^8$, then by the rules for logarithms we have

$$\ln y = \ln[e^x(3x-4)^8] = \ln e^x + \ln(3x-4)^8 = x + 8\ln(3x-4).$$

Differentiating both sides of the above equation using the chain rule we have

$$\begin{aligned} \frac{1}{y}\frac{d}{dx}y &= 1 + 8\frac{1}{3x-4}\frac{d}{dx}(3x-4) \\ \frac{1}{y}y' &= 1 + \frac{24}{3x-4}. \end{aligned}$$

Solving for y' we have

$$y' = y\left(1+\frac{24}{3x-4}\right) = e^x(3x-4)^8\left(1+\frac{24}{3x-4}\right).$$

(b) As above, we have

$$\ln y = 3\ln x + 4\ln(x-3) - 4\ln(x+4).$$

Differentiating both sides, we have

$$\frac{y'}{y} = \frac{3}{x} + \frac{4}{x-3} - \frac{4}{x+4}.$$

Solving for y' we have

$$y' = \frac{x^3(x-3)^4}{(x+4)^4}\left(\frac{3}{x} + \frac{4}{x-3} - \frac{4}{x+4}\right).$$

(c) We already know how to differentiate this function without logarithmic differentiation. Recall that we can write

$$y = 10^x = (e^{\ln 10})^x = e^{(\ln 10)x} \implies y' = e^{(\ln 10)x}\frac{d}{dx}(\ln 10)x = e^{(\ln 10)x}(\ln 10) = (\ln 10)10^x.$$

We should get the same result by logarithmic differentiation. We have

$$y = 10^x \implies \ln y = \ln(10^x) = x\ln 10 \implies \frac{y'}{y} = \ln 10 \implies y' = (\ln 10)y = (\ln 10)10^x.$$

Which method do you think is easier?

(d) We could differentiate this function by a method similar to the first method in the previous answer. On the other hand, by logarithmic differentiation, by the product rule we have

$$y = x^{1/x} \implies \ln y = \frac{1}{x}\ln x \implies \frac{y'}{y} = -x^{-2}\ln x + (x^{-1})^2 \implies y' = x^{1/x}\left(-\frac{\ln x}{x^2} + \frac{1}{x^2}\right).$$

8. We have $P(x) = R(x) - C(x) = 300\ln(x+1) - 2x$. Differentiating, $P'(x) = 300/(x+1) - 2$. Stationary points are where $300/(x+1) - 2 = 0$; solving for x gives $300 = 2(x+1) = 2x+2$ which implies $298 = 2x$ or $x = 149$. To check the nature of the stationary point we use the second derivative test: $P''(x) = -300/(x+1)^2$ is always negative, so the function is always concave down, so the stationary point must be a maximum. In summary, profit is maximize when $x = 149$.

9. Recall that revenue is price times number of units produced/sold, so $R(x) = px = (45/\ln x)x$. Simplifying slightly we have $R(x) = 45x/\ln x$. Then by the quotient rule the marginal revenue is

$$R'(x) = \frac{(\ln x)45 - 45x/x}{(\ln x)^2} = \frac{45\ln x - 45}{(\ln x)^2}.$$

When $x = 20$, the marginal revenue is $R'(20) = (45\ln 20 - 45)/(\ln 20)^2 = 10.0071$.

To find whether revenue is ever maximized, we first need to find levels of production at which the marginal revenue is 0: $R'(x) = 0$ implies $(45\ln x - 45)/(\ln x)^2 = 0$ implies $45\ln x - 45 = 0$ implies $\ln x = 1$ implies $x = e$, Euler's constant. To determine the nature of the stationary point, we could differentiate again and use the second derivative test. However, the calculations are tedious, so let's just try to use the first derivative test. Recall that $\ln x$ is increasing, so $45\ln x - 45$ is increasing. It follows that $45\ln x - 45$ is positive when $x > e$, negative when $x < e$. The denominator of $R'(x)$ is always positive, so it follows that $R(x)$ decreases to $x = e$ and then increases thereafter. By the first derivative test, there is a global minimum at $x = e$.

Furthermore, revenue is never maximized because marginal revenue is always positive to the right of $x = e$, i.e., we can always increase revenue by increasing production. Such a situation is unrealistic in practice, so our model is unreasonable for x larger than some value. To find that value we should investigate the assumptions that went into the model and see which of them breaks down for large x.