MATH 103 Problem Set 8 Solutions DRAFT

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1. The differential equation $P'(t) = kP(t)$ has solution $P(t) = P_0e^{kt}$. In this case $k = 0.03$ so we have $P(t) = P_0e^{0.03t}$. Furthermore, we have $P_0 = P(0)$ so we have $P(t) = 2.5e^{0.03t}$. At this point, it would be a good idea to verify that the function $P(t) = 2.5e^{0.03t}$ satisfies both the differential equation $P'(t) = 0.03P(t)$ and the initial condition $P(0) = 2.5$.

   If that formula holds over the time interval in question, we have the population in the year 2000 is $P(0) = 2.5e^{0.03(0)} = 2.5e^0 = 2.5 \times 1 = 2.5$; the population in the year 2003 is $P(3) = 2.5e^{0.03(3)} = 2.5e^{0.09} = 2.75$ million; and the population in the year 2006 is $P(6) = 2.5e^{0.03(6)} = 3$ million or so.

   We could do a complicated calculation involving derivates and logarithms to figure out how fast the population is growing when it reaches 4 million in size, but it's easier to use the differential equation. We have $P'(t) = 0.03P(t) = 0.03(4) = 0.12$ million, or about $120,000$ people per year. Notice that we didn't have to figure out what the relevant value of t is in order to answer the question.

2. Assuming that the population of the world grows at a rate proportional to its size means using an exponential model. (In detail, it means that population satisfies the differential equaton $P'(t) = kP(t)$ which implies that $P(t) = P_0e^{kt}$ for some P_0 and k, and the latter expression for $P(t)$ is the main thing.)

   For this problem, let's set $t = 0$ to mean January 1, 1993, and we measure time in years, so $t = 1$ is January 1, 1994, and January 1, 1998 is $t = 5$. The population at $t = 0$ is 5.51 (units are billions of people), so we have $P(t) = 5.51e^{kt}$. Now the only thing we need to know to determine $P(t)$ completely is the value of k, which we can find from the other bit of information we have. We know that on the one hand $P(5) = 5.88$, and on the other $P(t) = 5.51e^{k\cdot 5}$. That gives us an equation which we solve for k:

   $$5.88 = 5.51e^{k\cdot 5} \implies 5k = \ln\left(\frac{5.88}{5.51}\right) \implies k = \frac{1}{5}\ln\left(\frac{5.88}{5.51}\right) = 0.0130.$$

   Our model for the population of the earth is $P(t) = 5.51e^{0.0130t}$ where t is the number of years since January 1, 1993.

   Now we can use the model to answer the questions that were given. (a) The population is 6.5 billion means $6.5 = P(t) = 5.51e^{0.0130t}$; solving for t gives $t = 12.71$, or sometime in late September 2005. (b) The population is 8 billion means $8 = 5.51e^{0.0130t}$ which implies $t = (1/0.0130)\ln(8/5.51) = 28.68$, or sometime in the second half of 2021. (c) The population is 10 billion when $t = (1/0.0130)\ln(10/5.51) = 45.8$ or sometime in 2038.

3. (a) Assuming continuous compounding, $A(t) = 3000e^{0.045t}$.
   (b) The differential equation is $A'(t) = 0.045A(t)$. You should check that the above function $A(t)$ satisfies that equation. Note that the differential equation depends on the interest rate but not on the principal.
   (c) $A(0) = 3000$, the amount of the initial deposit. Note that $A(0)$ depends on the principal but not the interest rate.
   (d) After three years, the amount in the account is $A(3) = 3000e^{0.045\cdot 3} = 3433.61$.
   (e) The balance reaches 7000 when $7000 = A(t)$. We use the formula for $A(t)$ and solve the resulting equation: $7000 = 3000e^{0.045t}$ which implies $t = (1/0.45)\ln(7000/3000) = 18.8$, i.e., nearly 19 years into the future.

(f) The easiest way to answer that question is to use the differential equation. The balance is growing at a rate of $A'(t) = 0.045A(t) = 0.045(7000) = 315$ dollars per year. Alternatively, you could calculate $A'(18.8)$: $A(t) = 3000e^{0.045t}$ so $A'(t) = 3000 \cdot 0.045e^{0.045t} = 135e^{0.045t}$ so $A'(18.8) = 135 \cdot e^{0.045 \cdot 18.8} = 314.59$. There is some loss in accuracy due to rounding $t = 18.8$.

4. Suppose that the initial investment is P. Then $A(t) = Pe^{0.04t}$. The doubling time is when $A(t) = 2P$, i.e., $2P = Pe^{0.04t}$. Solving for t we have $2 = e^{0.04t}$ which implies $t = (1/0.04)\ln 2 = 17.3$, just over 17 years. The tripling time is when $A(t) = 3P$ which leads to $t = (1/0.04)\ln 3 = 27.5$ years.

   At 10% the times become $t = (1/0.10)\ln 2 = 6.9$ years to double and $t = (1/0.10)\ln 3 = 11.0$ years to triple.

5. Assuming continuous compounding we have the value of the house is $A = Pe^{rt}$. Filling in $A = 180,000$, $P = 135,000$, and $t = 3$ years, we have the equation

   $$180,000 = 135,000e^{r(3)} \implies \frac{180,000}{135,000} = e^{3r} \implies 3r = \ln\left(\frac{180}{135}\right) \implies r = \frac{1}{3}\ln\left(\frac{180}{135}\right).$$

   Pressing the correct sequence of buttons on my calculator gives $r = 9.6\%$. (If you're familiar with the formula for annual compounding, you should be able to show that the corresponding rate for annual compounding is just slightly over 10%.)

6. The rate of return for large cap (Standard & Poor, S&P) stocks is found by solving

   $$517 = 1e^{r(65)} \implies 65r = \ln 517 \implies r = \frac{1}{65}\ln 517 = 0.0961$$

   about 9.6%, assuming continuous compounding. For small cap stocks we have $r = \ln(1277)/65 = 0.11$, about 11%. Note that a small difference in the interest rate over the period resulted in a future value which was almost double.

   (If you're interested in financial math, you might like to try the above calculations using annual compounding instead of continuous compounding. If we use annual compounding instead of continuous compounding, in the first case we get $(1+r)^{65} = 517$ which implies $\ln(1+r) = (1/65)\ln 517$ which implies $r = e^{ln(517)/65} - 1 = 10.0\%$, and in the second case we get $r = 11.63\%$, off by about half a percent. The annual compounding numbers are the ones usually quoted when talking about the stock market. Which model do you think is better for the stock market, annual compounding or continuous compounding?)

   As for your investment, at 9.6% we have $A = 5000e^{0.096(40)} = 233,000$ or so, and at 11% we have $A = 5000e^{0.11(40)} = 407,000$. Moral of the story: start investing when you're young, and try to squeeze an extra percentage point or two out of your investment, e.g., by keeping management fees low.

7. We have $A(t) = 10,000e^{0.08t}$. At the beginning of the second year the investment is worth $A(1) = 10,000e^{0.08 \cdot 1} = 10,832.87$. At the end of the second year the investment is worth $A(2) = 11,735.11$. The interest earned during the second year is $A(2) - A(1) = 902.24$ (rounding errors may change that value slightly).

8. The amount of cesium-137 we have is $A(t) = A_0e^{-0.023t}$ where A_0 is the initial amount and t is measured in years. Note the negative sign, which is needed because it is a case of decay, not growth. To find the half life, we can approach the problem in two different ways. In the first method, we need to find when $A(t) = A_0/2$, half the original amount. Then we have $A_0/2 = A_0e^{-0.023t}$ which implies $t = (-1/0.023)\ln(1/2) = (1/0.023)\ln 2 = 30.14$, about 30 years. Note that the actual value of A_0 is not important because it cancels out of the equation before we find t.

   The other method is to re-write $A(t)$ in terms of 2^x rather than e^x. We have $2 = e^{\ln 2}$ so $2^{x/\ln 2} = (e^{\ln 2})^{x/\ln 2} = e^x$ so $A(t) = A_0e^{-0.023t} = A_02^{-0.023t/\ln 2} = A_02^{-0.0332t}$. Half life is when $2^{-0.0332t} = 1/2 = 2^{-1}$, i.e., $t = 1/0.0332 = 30.14$, about 30 years, in agreement with the previous method.

9. Again, we could do this in two different ways. The first way is to use the function 2^x. We have that the amount of carbon-14 at time t in years is $A(t) = A_02^{-t/5730}$. We need $A(t) = 0.34A_0$, 34% of its original value, in which case $0.34A_0 = A_02^{-t/5730}$ which implies $0.34 = 2^{-t/5730}$. Taking the natural logarithm of both sides gives $\ln 0.34 = (-t/5730)\ln 2$ which implies $t = -5730(\ln 0.34)/\ln 2) = 8918$ years.

The second way is to use the exponential function e^x, so $A(t) = A_0 e^{-kt}$ where k is the decay constant. When we used 2^x the decay constant was easy to see, it was just $1/5730$, the reciprocal of the half life. In this case, we have to solve for it. We have $A(5730) = A_0/2$ on the one hand and $A_0 e^{-k\cdot 5730}$ on the other. Solving for k we have $A_0/2 = A_0 e^{-k\cdot 5730}$ which implies $1/2 = e^{-5730k}$ which implies $k = (-1/5730)\ln(1/2) = (1/5730)\ln(2) = 0.00012097$ (I used extra decimal points, otherwise the loss of accuracy would be too great). Now we need to find t such that $A(t) = 0.34A_0$. On the other hand we have $A(t) = A_0 e^{-0.00012097t}$; solving for t gives $0.34A_0 = A_0 e^{-0.00012097t}$ which implies $t = -\ln(0.34)/0.00012097 = 8918$ years.