MATH 103 Sample Final Exam Solutions DRAFT

Edward Doolittle

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1. The definition of derivative says

$$f'(1) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

In this case we have

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{1+h+3} - \sqrt{1+3}}{h} = \lim_{h\to 0} \frac{\sqrt{h+4} - \sqrt{4}}{h}.$$

(NOTE: $\sqrt{x+3}$ *is not the same as* $\sqrt{x} + \sqrt{3}$!!) The best way to evaluate the above limit is using the conjugate radical:

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{h+4} - \sqrt{4}}{h} \frac{\sqrt{h+4} + \sqrt{4}}{\sqrt{h+4} + \sqrt{4}} = \lim_{h\to 0} \frac{(h+4) - (4)}{h(\sqrt{h+4} + \sqrt{4}} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3}}.$$

Cancelling the hs in the numerator and denominator and then letting h approach 0 gives

$$f'(1) = \lim_{h\to 0} \frac{1}{\sqrt{h+4} + \sqrt{4}} = \frac{1}{\sqrt{0+4} + \sqrt{4}} = \frac{1}{2+2} = \frac{1}{4}.$$

You can check the result using the power rule and the chain rule.

2. On the one hand we have

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} e^{x^2-1} = e^{(-2)^2-1} = e^3.$$

On the other hand we have $f(-2) = e^2$. Since $\lim_{x\to -2} f(x) \neq f(-2)$, the function $f(x)$ is *not* continuous at $x = -2$.

3. (a) Factoring,

$$\lim_{x\to 2} \frac{x^2+3x-10}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x+5)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+5}{x+2} = \frac{7}{4}.$$

(b) Dividing through by the highest power of x occurring in the expression we have

$$\lim_{x\to\infty} \frac{x^2-4x+3}{5x^2+6} = \lim_{x\to\infty} \frac{(x^2-4x+3)/x^2}{(5x^2+6)/x^2} = \lim_{x\to\infty} \frac{1-4/x+3/x^2}{5+6/x^2} = \frac{1-0+0}{5+0} = \frac{1}{5}.$$

4. (a) By the product rule and the chain rule,

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}(x^2+1)^{3/2}\right)(2x-3) + (x^2+1)^{3/2}\left(\frac{d}{dx}(2x-3)\right) \\ &= \frac{3}{2}(x^2+1)^{1/2}\left(\frac{d}{dx}(x^2+1)\right)(2x-3) + (x^2+1)^{3/2}(2) \\ &= 3x(x^2+1)^{3/2}(2x-3) + 2(x^2+1)^{3/2}. \end{aligned}$$

(b) By the quotient rule,

$$g'(t) = \frac{(4t^2+1)\frac{d}{dt}(3t+2) - (3t+2)\frac{d}{dt}(4t^2+1)}{(4t^2+1)^2} = \frac{(4t^2+1)(3) - (3t+2)(8t)}{(4t^2+1)^2}.$$

Simplification is possible but is not really helpful for this question.

(c) By the sum and constant multiple rules,

$$y' = 2\frac{d}{dx}e^x + \frac{d}{dx}e^3 - \frac{d}{dx}\ln(1+x^2).$$

The first term is the derivative of an exponential. The second term is the derivative of a constant (e^3 is a constant), so is zero. The last term is the derivative of a logarithm with the chain rule. Altogether,

$$y' = 2e^x - \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) = 2e^x - \frac{2x}{1+x^2}.$$

(d) By the chain rule, the first derivative is

$$f'(x) = e^{x^2+1}\frac{d}{dx}(x^2+1) = 2xe^{x^2+1}.$$

To find the second derivative, we must apply the product rule and then the chain rule as above:

$$f''(x) = 2e^{x^2+1} + 2x\frac{d}{dx}e^{x^2+1} = 2e^{x^2+1} + 2xe^{x^2+1}2x = (2+4x^2)e^{x^2+1}.$$

5. To find the equation of the tangent line we need to find a point on the line and the slope of the line. The point is given: $(x_0, y_0) = (1, 3)$. The slope m is the derivative of the function at x_0, i.e.,

$$m = f'(x_0) = \frac{1}{2}(x_0^2+8)^{-1/2}\cdot 2x_0 = \frac{1}{2}(1^2+8)^{-1/2}\cdot 2(1) = \frac{1}{9^{1/2}} = \frac{1}{3}.$$

The point-slope form of the line is

$$y - 3 = \frac{1}{3}(x-1).$$

Simplification is possible but is not necessary.

6. The critical points of the function (also called the stationary points) are the x-values where $f'(x) = 0$. Taking the derivative we have

$$f'(x) = -e^{2x} + (1-x)\cdot 2e^{2x} = (1-2x)e^{2x}.$$

Simplification really is necessary in this question. Now we solve the equation $f'(x) = 0$:

$$f'(x) = 0 \implies (1-2x)e^{2x} = 0 \implies 1-2x = 0 \implies x = \frac{1}{2}.$$

The only critical point is at $x = 1/2$. Note that in the above we used the fact that e^{2x} is never zero, so we can divide the equation by that term.

7. We find critical points as in the previous problem:

$$f'(x) = 3x^2 - 6x = 0 \implies 3(x-2)x = 0 \implies x = 0, 2.$$

To determine the nature of the extreme point we use the second derivative test: $f''(x) = 6x-6$ so $f''(0) = -6 < 0$, relative maximum at $x = 0$; $f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$, relative minimum at $x = 2$. To locate the points on the graph we find the corresponding y-values: when $x = 0$, $y = f(0) = 0^3 - 3(0)^2 + 4 = 4$, so the relative maximum is at the point $(0, 4)$ on the graph. When $x = 2$, $y = f(2) = 2^3 - 3(2)^2 + 4 = 8 - 12 + 4 = 0$, so the relative minimum is at the point $(2, 0)$.

Inflection points are where $f''(x) = 0$, i.e., $6x - 6 = 0$, i.e., $x = 1$. The corresponding y-value is at $y = f(1) = 1^3 - 3(1)^2 + 4 = 2$, so the inflection point on the graph is $(1, 2)$.

Assembling all the above information gives [INSERT GRAPH HERE]

8. It helps to draw a sketch. Let the sides of the garden parallel to the fence be of length x feet, and the sides of the garden perpendicular to the fence be of length y feet. The constraint is that the area of the garden is 120 square feet, so we have $xy = 120$. The object is to minimize the cost of the garden which is $C = 15y + 15x + 15y + 10x = 30y + 25x$. We can write the objective in terms of a single variable by using the constraint to eliminate one variable. We have $y = 120/x$ so

$$C(x) = 30\frac{120}{x} + 25x = 3600x^{-1} + 25x.$$

To minimize, we take the first derivative to obtain

$$C'(x) = -3600x^{-2} + 25.$$

The minimum should occur where $C'(x) = 0$, i.e., $-3600x^{-2} + 25 = 0$, i.e., $3600x^{-2} = 25$, i.e., $144 = x^2$, i.e., $x = \pm 12$. Since a negative value for x doesn't make sense we throw it away and conclude that $x = 12$ is the only stationary (critical) point for the function. Testing the nature of the point be the second derivative test we have $C''(x) = 7200x^{-3}$ so $C''(12) = 7200(12)^{-3} > 0$ and the stationary point is a (local) minimum.

Finally, to figure out the dimensions of the garden, we use the constraint to obtain $y = 120/x = 120/12 = 10$. The garden should be 12 feet long in the direction parallel to the fence, and 10 feet long in the direction perpendicular to the fence.

9. The demand function $p = f(x)$ passes through the point $(p,x) = (50,200)$. We are also told that if x increases by 1 then p must decrease by 2, so the slope of the demand function is rise/run $= -2/1 = -2$. The point slope form of the linear demand function is

$$p - 50 = -2(x - 200) \implies p = 450 - 2x.$$

(It is helpful to solve for p in this question.) The revenue function is then

$$R(x) = x \cdot p = x(450 - 2x) = 450x - 2x^2.$$

The cost function is

$$C(x) = 5000 + 36x,$$

i.e., the fixed cost plus the cost per person multiplied by the number of people. The profit function is then

$$P(x) = R(x) - C(x) = 450x - 2x^2 - (5000 + 36x) = 414x - 2x^2 - 5000.$$

Profit is maximized when $P'(x) = 414 - 4x = 0$ which implies $x = 103.50$. Checking that that is a maximum rather than a minimum, we have $P''(103.50) = -4 < 0$, so a local maximum. The tour operator should charge \$103.50 per seat to maximize profit.

10. The purchase value of the computer is $v(0) = 2000e^{-0.35(0)} = 2000e^0 = 2000 \cdot 1 = 2000$ dollars. Half the purchase value is then 1000 dollars. The value of the computer is 1000 dollars when $v(t) = 1000$ which implies $1000 = 2000e^{-0.35t}$ which implies $e^{-0.35t} = 0.5$. Taking the logarithm of both sides we have $-0.35t = \ln(0.5)$ which implies $t = \ln(0.5)/(-0.35) = 1.9804$, or very close to 2 years.

at my home computer.

11. (a) By the sum and constant multiple rules for integrals,

$$\int \left(\frac{2}{x} + 8x^3 - 6e^x\right)\,dx = 2\int \frac{1}{x}\,dx + 8\int x^3\,dx - 6\int e^x\,dx.$$

Recall that to find the integral of a power of x, you add 1 to the power and divide by the new power, except if the power of x is -1, in which case the answer is the natural logarithm. Also recall that the integral of e^x is just e^x again. So the answer is

$$2\ln x + 8\frac{x^4}{4} - 6e^x + C = 2\ln x + 2x^4 - 6e^x + C$$

where the simplification in the last step was not necessary, but the constant C is necessary.

(b) By the Fundamental Theorem of Calculus, we have

$$\int_0^{\ln 3} 9e^{-3x}\,dx = \int 9e^{-3x}\,dx\Big|_0^{\ln 3}.$$

Evaluating the indefinite integral,

$$\int 9e^{-3x}\,dx = 9\int e^{-3x}\,dx = 9\frac{e^{-3x}}{-3} + C = -3e^{-3x} + C.$$

You should check the above result by differentiating $-3e^{-3x} + C$. You should get $9e^{-3x}$. Therefore

$$\int_0^{\ln 3} 9e^{-3x}\,dx = \left. -3e^{-3x}\right|_0^{\ln 3} = (-3e^{-3(\ln 3)}) - (-3e^{-3(0)}) = -33^{-3} - (-3(1)) = -\frac{1}{9} + 3 = \frac{26}{9}.$$

(c) This is an example of integration by substitution. Let $u = 1 + x^4$. Then $du/dx = 4x^3$ which implies $du/4 = x^3\,dx$. Making the substitution we have

$$\int \frac{x^3}{1+x^4}\,dx = \int \frac{1}{u}\,\frac{du}{4} = \frac{1}{4}\ln u + C.$$

Now reversing the substitution we have

$$\frac{1}{4}\ln u + C = \frac{1}{4}\ln(1+x^4) + C,$$

the answer to the question.

(d) This is a more difficult example of integration by substitution. This time, let $u = 1 + e^x$, the expression under the square root sign. Then $du = e^x\,dx$ so we can write

$$\int e^x\sqrt{1+e^x}\,dx \int \sqrt{1+e^x}\,e^x\,dx = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

You should check by differentiating the above expression; you should get the integrand of the question.