Math 111 Term Test 1 Solutions

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1. (a) By the chain rule, $y' = e^{e^x} \frac{d}{dx} e^x = e^{e^x} e^x$.

   (b) By implicit differentiation,
$$\frac{d}{dx} e^{x^2 y} = \frac{d}{dx}(x+y)$$
$$e^{x^2 y} \frac{d}{dx}(x^2 y) = 1 + y'$$
$$e^{x^2 y}(2xy + x^2 y') = 1 + y'.$$
   Solving for y',
$$x^2 e^{x^2 y} y' - y' = 1 - 2xye^{x^2 y}$$
$$y' = \frac{1 - 2xye^{x^2 y}}{x^2 e^{x^2 y} - 1}.$$

   (c) By the fundamental theorem of calculus, $\frac{d}{dt} \int_1^t e^{x^2+x+1}\, dx = e^{t^2+t+1}$.

   (d) Write $y = (1 + xe^{-2x})^{1/2}$. Then by the chain rule and product rule,
$$y' = \frac{1}{2}(1 + xe^{-2x})^{-1/2} \frac{d}{dx}(1 + xe^{-2x}) = \frac{1}{2}(1 + xe^{-2x})^{-1/2}(e^{-2x} + xe^{-2x}(-2)).$$

2. (a) Let $u = -3x$. Then $du = -3dx$ so $dx = du/(-3)$ and we have
$$\int e^{-3x}\, dx = \int e^u \frac{du}{-3} = -\frac{1}{3} \int e^u\, du = -\frac{1}{3} e^u + C = -\frac{1}{3} e^{-3x} + C.$$
   Evaluating the definite integral,
$$\int_0^5 e^{-3x}\, dx = -\frac{1}{3} e^{-3x} \Big|_0^5 = -\frac{1}{3}(e^{-15} - 1).$$

   (b) Dividing the integrand through by e^x,
$$\int \frac{e^x + 1}{e^x}\, dx = \int \frac{e^x}{e^x} + \frac{1}{e^x}\, dx = \int 1 + e^{-x}\, dx = x - e^{-x} + C.$$

   (c) $\int_0^1 xe^{-x^2}\, dx$ Let $u = -x^2$. Then $du = -2x\, dx$ so $x\, dx = -du/2$ and
$$\int xe^{-x^2}\, dx = \int e^u \frac{-du}{2} = -\frac{1}{2} \int e^u\, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C.$$
   Evaluating the definite integral,
$$\int_0^1 xe^{-x^2}\, dx = -\frac{1}{2} e^{-x^2} \Big|_0^1 = -\frac{1}{2}(e^{-1} - 1).$$

(d) Let $u = e^{-x}$. Then $du = -e^{-x}\,dx$ so $e^{-x}\,dx = -du$ and

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = \int \frac{e^{-x}\,dx}{1+(e^{-x})^2} = \int \frac{-du}{1+u^2} = -\arctan u + C = -\arctan(e^{-x}) + C.$$

3. Differentiating, $g'(x) = e^{2x-x^2}(2-2x)$. Setting the first derivative equal to 0, $e^{2x-x^2}(2-2x) = 0$. Dividing through by e^{2x-x^2} (which is possible because exponentials are never equal to 0 and in fact are always positive) gives the linear equation $2-2x = 0$ which implies $x = 1$. So a candidate for the absolute maximum value of the function occurs when $x = 1$. To check that that is the absolute maximum value, use the first derivative test. When $x < 1$, $2-2x > 0$ so $g'(x) = e^{2x-x^2}(2-2x) > 0$ so the function is increasing. When $x > 1$, $2-2x < 0$ so $g'(x) = e^{2x-x^2}(2-2x) < 0$ so the function is decreasing. The function increases to $x = 1$ and decreases thereafter, so the absolute maximum value of the function must occur when $x = 1$. The value is $g(1) = e^{2-1} = e$.

4. Differentiating again by the product rule and chain rule, $g''(x) = e^{2x-x^2}(2-2x)^2 + e^{2x-x^2}(-2) = e^{2x-x^2}(4-8x+4x^2-2) = e^{2x-x^2}(4x^2-8x+2)$. Similar to the previous problem, $g''(x) = 0$ implies

$$2x^2 - 4x + 1 = 0. \tag{1}$$

By the quadratic formula,

$$x = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{2}}{2}.$$

Calculator says $x_1 = 0.3$ and $x_2 = 1.7$ approximately. Furthermore $2x_1^2 - 4x_1 + 1 = 0$ and $2x_2^2 - 4x_2 + 1 = 0$ because x_1 and x_2 satisfy the quadratic equation (1); dividing by 2 and rearranging gives $2x_1 - x_1^2 = 1/2$ and $2x_2 - x_2^2 = 1/2$. Therefore $g(x_1) = e^{2x_1 - x_1^2} = e^{1/2}$ and $g(x_2) = e^{2x_2 - x_2^2} = e^{1/2}$; $g(x_1) = g(x_2) = 1.6$ approximately, so the inflection points are $(x_1, g(x_1)) = (0.3, 1.6)$ and $(x_2, g(x_2)) = (1.7, 1.6)$ to one decimal point of accuracy.

5. The only information still required is the intervals where the function is concave up/down. The sign of $g''(x)$ is the same as the sign of $2x^2 - 4x + 1$. Since the coefficient of x^2 is positive, $2x^2 - 4x + 1$ is negative between the two roots and positive elsewhere. (See figure 1, which isn't a required part of the answer, but is just there to give you a sense of how $2x^2 - 4x + 1$ changes sign; you can also use algebra to explore the sign changes.) Therefore

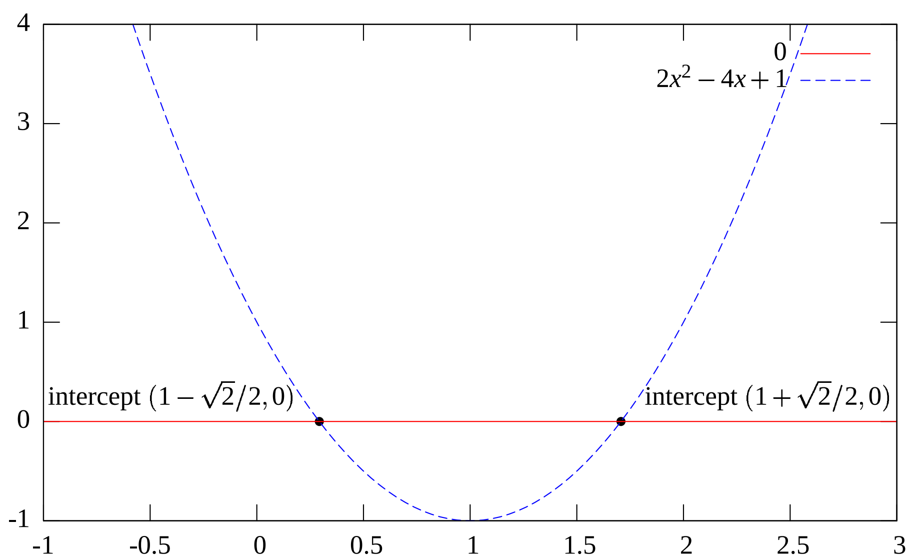

Figure 1: Graph of $2x^2 - 4x + 1$

$g''(x) > 0$ on the intervals $(-\infty, 0.3)$ and $(1.7, \infty)$, so the function $g(x)$ is concave up on those intervals; and $g''(x) < 0$ on the interval $(0.3, 0.7)$, so the function $g(x)$ is concave down on those intervals.

Assembling all the information we have gathered gives the graph in figure 2.

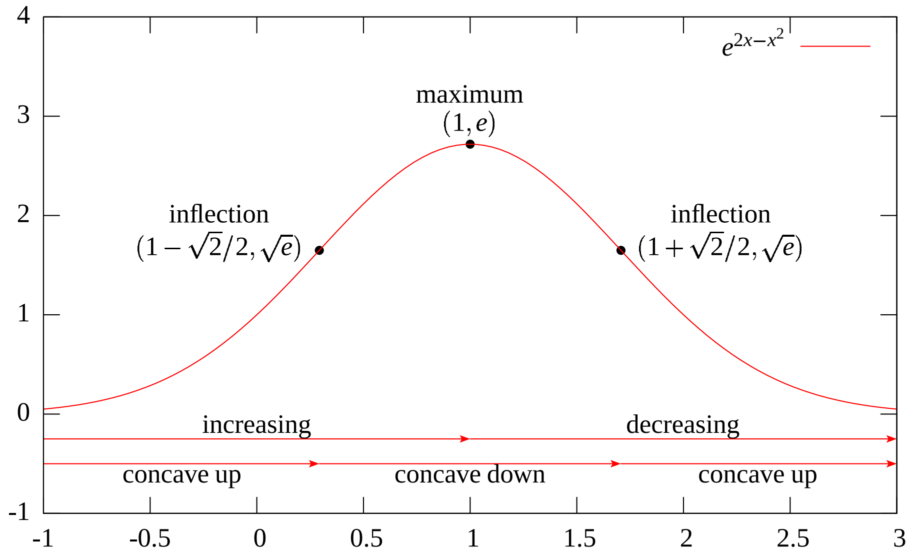

Figure 2: Graph of e^{2x-x^2}

6. Calculating the derivative, $f'(x) = 1 + e^x$. (Note that $f'(x) = 1 + e^x > 1 > 0$ so f is increasing everywhere, so is 1–1, so is invertible.) We also need to know $f^{-1}(4)$. Guessing, $f(0) = 3 + 0 + e^0 = 3 + 1 = 4$, so $f^{-1}(4) = 0$. Therefore by the formula for the derivative of an inverse function, $(f^{-1})'(4) = 1/f'(f^{-1}(4)) = 1/f'(0) = 1/(1+1) = 1/2$.

7. Let's warm up by proving a simpler result first.

   (a) Theorem: $e^x > 1$ for $x > 0$. Proof: Form the function $f(x) = e^x - 1$. Then $f'(x) = e^x > 0$ for all x so f is increasing for $x > 0$. Furthermore $f(0) = e^0 - 1 = 1 - 1 = 0$ so f increases above 0 for $x > 0$, i.e., $f(x) = e^x - 1 > 0$ for $x > 0$, i.e., $e^x > 1$ for $x > 0$.

   (b) Now we are ready for the given problem. Theorem: $e^x > x + 1$ for $x > 0$. Proof: We prove this in a way similar to that of the previous result. Let $g(x) = e^x - (x+1)$. Then $g'(x) = e^x - 1$, and we know that $e^x - 1 > 0$ for $x > 0$ from the previous result, so $g'(x) > 0$ for $x > 0$ which implies that $g(x)$ is increasing for $x > 0$. Also, $g(0) = e^0 - 0 - 1 = 1 - 1 = 0$, so $g(x)$ is increasing above 0 for $x > 0$, i.e., $g(x) = e^x - (x+1) > 0$ for $x > 0$, i.e., $e^x > x + 1$ for $x > 0$.