

Math 111 Practice Midterm Test 2 Solutions DRAFT

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1. (a) Exponentiating both sides of the equation, then adding 2 to both sides, then taking the natural logarithm of both sides,

$$\begin{aligned}e^x - 2 &= e^3 \\e^x &= e^3 - 2 \\x &= \ln(e^3 - 2)\end{aligned}$$

which is 2.8951 to four decimal places.

- (b) Use the cancellation equations $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ to simplify the given equation. You could use a calculator for the right hand sides, but since your calculator likely doesn't have a cot button you might get stuck at the last step. So instead, you should find the exact value of the right hand side at each step by geometry. Taking \cos^{-1} of both sides,

$$\begin{aligned}\cos^{-1}(\cos(\tan^{-1}(\sin(\cot^{-1}x)))) &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ \tan^{-1}(\sin(\cot^{-1}x)) &= \frac{\pi}{4}\end{aligned}$$

by the geometry of the $1, 1, \sqrt{2}$ right triangle. Now taking \tan of both sides,

$$\begin{aligned}\tan(\tan^{-1}(\sin(\cot^{-1}x))) &= \tan\left(\frac{\pi}{4}\right) \\ \sin(\cot^{-1}x) &= 1\end{aligned}$$

again by the geometry of the $1, 1, \sqrt{2}$ right triangle, and taking \sin^{-1} of both sides,

$$\begin{aligned}\sin^{-1}(\sin(\cot^{-1}x)) &= \sin^{-1}(1) \\ \cot^{-1}x &= \frac{\pi}{2}\end{aligned}$$

by the geometry of the degenerate $0, 1, 1$ right triangle. Now taking \cot of both sides,

$$\begin{aligned}\cot(\cot^{-1}x) &= \cot\left(\frac{\pi}{2}\right) \\ x &= \frac{\cos(\pi/2)}{\sin(\pi/2)} = 0\end{aligned}$$

again by the geometry of the degenerate $0, 1, 1$ right triangle. (Question: is $x = 0$ the only solution? Another question: if you use a calculator, does it matter what angle mode (degree, radian, gradient) it is in?)

2. (a) It is easiest to simplify the logarithm before differentiating:

$$f(x) = \log_{10}x - \log_{10}(x-1) = \frac{\ln x}{\ln 10} - \frac{\ln(x-1)}{\ln 10}$$

so

$$f'(x) = \frac{1}{\ln 10} \left(\frac{1}{x} - \frac{1}{x-1} \right).$$

(b) By the rule for the derivative of \tan^{-1} and the chain rule,

$$g'(x) = \frac{1}{1+x^2} + \frac{1}{1+(1/x)^2} \cdot \frac{1}{x^2} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

(Question: does this mean that g is constant? Careful!)

3. Taking the logarithm of both sides and applying a law of logarithms,

$$\ln y = \ln(\ln x)^{\cos x} = \cos x \ln \ln x.$$

By implicit differentiation, and the product and chain rules applied to the left hand side,

$$\frac{y'}{y} = -\sin x \ln \ln x + \cos x \frac{1}{\ln x} \cdot \frac{1}{x}.$$

Multiplying both sides by $y = (\ln x)^{\cos x}$,

$$y' = (\ln x)^{\cos x} \left(-\sin x \ln \ln x + \frac{\cos x}{x \ln x} \right).$$

4. (a) By the substitution $u = \ln x$, $du = (1/x) dx$,

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

(b) The indefinite integral is easy if you have memorized the derivatives of the inverse trig functions:

$$\int \frac{4}{t^2+1} dt = 4 \tan^{-1} t + C.$$

(Alternatively, you could use the trig substitution $t = \tan \theta$.) Now the definite integral can be evaluated:

$$\int_0^1 \frac{4}{t^2+1} dt = 4 \tan^{-1} t \Big|_0^1 = 4 \tan^{-1} 1 - 4 \tan^{-1} 0 = 4 \frac{\pi}{4} = \pi.$$

To four decimal places, Archimedes' constant is 3.1416.

5. (a) Since $\ln 1 = 0$ and $\sin \pi \cdot 1 = 0$ the limit is of the form $0/0$. By L'Hôpital's rule,

$$\lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x} = \lim_{x \rightarrow 1} \frac{1/x}{\pi \cos \pi x} = \frac{1/1}{\pi \cos \pi} = -\frac{1}{\pi}.$$

(b) The limit is of the form 0^0 so we first apply the logarithm function to turn the limit into a product. Let $L = \lim_{x \rightarrow 0^+} x^{x^2}$. Then because the natural logarithm function is continuous we can bring it inside the limit and we have

$$\ln L = \ln \lim_{x \rightarrow 0^+} x^{x^2} = \lim_{x \rightarrow 0^+} \ln x^{x^2} = \lim_{x \rightarrow 0^+} x^2 \ln x,$$

applying a law of logarithms in the last step. The limit is now of the form $0 \cdot \infty$, so we move one function into the denominator; x^2 is the simpler choice, and we have

$$\ln L = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x^2}.$$

That limit is of the form ∞/∞ so we can apply L'Hôpital's rule:

$$\ln L = \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0^+} -\frac{x^2}{2} = 0.$$

Solving for L , $L = e^L = e^0 = 1$.

6. First, it's a good idea to check that $(0, 2)$ is actually a point on the curve: $(2 + 0)e^{-0} = 2 \cdot 1 = 2$ so $(0, 2)$ really is a point on the curve. Differentiating by the product and chain rules,

$$y' = (1)e^{-x} + (2 + x)e^{-x}(-1) = (-1 - x)e^{-x}.$$

The slope of the tangent line at $x = 0$ is $m = y'(0) = (-1 - 0)e^{-0} = -1$. Therefore an equation of the tangent line in point slope form is

$$(y - y_0) = m(x - x_0) \implies (y - 2) = (-1)(x - 0).$$

There's no need to simplify the equation.

7. The easiest way to do this integral is by the substitution $u = 1 + \sec\theta$, $du = \sec\theta \tan\theta d\theta$,

$$\int \frac{\sec\theta \tan\theta}{1 + \sec\theta} d\theta = \int \frac{du}{u} = \ln|u| + C = \ln|1 + \sec\theta| + C.$$

Alternatively, you can write $\sec\theta = 1/\cos\theta$, $\tan\theta = \sin\theta/\cos\theta$, clear fractions, and make the substitution $u = \cos\theta$. You would then have to apply partial fractions (easy, but we haven't learned it) or complete the square in the denominator and apply another trig substitution.

8. Let $F(t) = \int_1^t \frac{e^s}{s} ds$. Then we have to differentiate the function $F(\sqrt{x})$ so we apply the chain rule:

$$\frac{d}{dx}F(\sqrt{x}) = F'(\sqrt{x}) \frac{1}{2}x^{-1/2}.$$

We find $F'(t)$ by the Fundamental Theorem of Calculus:

$$\frac{d}{dt} \int_1^t \frac{e^s}{s} ds = \frac{e^t}{t}.$$

Putting it all together,

$$\frac{d}{dx} \int_1^{\sqrt{x}} \frac{e^s}{s} ds = \frac{e^{\sqrt{x}}}{\sqrt{x}} \frac{1}{2}x^{-1/2}.$$

Simplification of the above result is possible, but not necessary.