Math 111 Midterm Test 2 Solutions

Edward Doolittle

November 29, 2005

1. (a) Taking the power of 2 of both sides of the equation,

$$2^{\log_2(x+1)} = 2^4 \implies x+1 = 16 \implies x = 15.$$

   (b) If you're going to use a calculator, it will help you follow along below if your calculator is in radian mode, although you'll still get the correct answer if it is in degree mode. Also, since there likely isn't an arccsc button on your calculator, you will need to rewrite the equation in terms of sin. Then the given equation implies

$$\frac{1}{\sin(\arccos x)} = 1.25 \implies \arccos x = \arcsin 0.8 = 0.9273 \implies x = \cos 0.9273 = 0.6000.$$

   If we're getting technical, there are really an infinite number of values $\arccos x$ could take to satisfy the equation, but most, if not all, of the other values are outside of the range of arccos and therefore irrelevant. You should draw some graphs to understand the situation.

   Alternatively, you could use a $3,4,5$ right triangle to solve the problem; how?

2. (a) It is best to rewrite in terms of the natural exponential function:

$$f(\theta) = e^{(\ln 10)\tan\theta} \implies f'(\theta) = e^{(\ln 10)\tan\theta} \cdot \ln 10 \sec^2\theta = (\ln 10)10^{\tan\theta}\sec^2\theta.$$

   (b) It is best to prepare the question first by simplifying the logarithm:

$$g(x) = \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\ln(x+a) - \frac{1}{2}\ln(x-a)$$

   so

$$g'(x) = \frac{1}{1+(x/a)^2}\left(\frac{1}{a}\right) + \frac{1}{2(x+a)} - \frac{1}{2(x-a)} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}.$$

   The above simplifications are not necessary.

3. Taking the logarithm of the function,

$$\ln y = \ln\sqrt[4]{\frac{x^2+1}{x^2-1}} = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1).$$

   Differentiating both sides (implicitly),

$$\frac{y'}{y} = \frac{1}{4(x^2+1)}2x - \frac{1}{4(x^2-1)}2x = \frac{x}{2}\left(\frac{1}{x^2+1} - \frac{1}{x^2-1}\right) = -\frac{x}{x^4-1}.$$

   Solving for y',

$$y' = -\frac{x}{x^4-1}\sqrt[4]{\frac{x^2+1}{x^2-1}}.$$

   The various simplifications above were again not necessary but I couldn't resist.

4. (a) A little algebra gives

$$\int \frac{1+x-x^2}{x^2}\,dx = \int x^{-2}+x^{-1}-1\,dx = -x^{-1}+\ln|x|-x+C.$$

Evaluating the definite integral,

$$\int_2^4 \frac{1+x-x^2}{x^2}\,dx = -\frac{1}{4}+\ln 4-4+\frac{1}{2}-\ln 2+2 = \frac{1}{4}-2+\ln(4/2) = \ln 2-\frac{7}{4}.$$

To four decimal places, the answer is -1.0569.

(b) Make the change of variables $u=\tan\theta$, $du=\sec^2\theta\,d\theta$ to obtain

$$\int 2^{\tan\theta}\sec^2\theta\,d\theta = \int 2^u\,du = \int e^{(\ln 2)u}\,du = \frac{1}{\ln 2}e^{(\ln 2)u}+C = \frac{2^{\tan\theta}}{\ln 2}+C.$$

Check by differentiating.

5. (a) The numerator and denominator both tend to 0 as $x\to 0$ so we can apply L'Hôpital's rule to obtain

$$\lim_{x\to 0}\frac{e^{4x}-1-4x}{x^2} = \lim_{x\to 0}\frac{4e^{4x}-4}{2x}.$$

The numerator and denominator of the above expression still both tend to 0 as $x\to 0$ so we can apply L'Hôpital's rule again to obtain

$$\lim_{x\to 0}\frac{4e^{4x}-4}{2x} = \lim_{x\to 0}\frac{16e^{4x}}{2} = 8.$$

(b) The limit is of the form 1^∞ so we take the logarithm and apply L'Hôpital's rule:

$$\ln L = \lim_{x\to 0^+}\ln(\cos x)^{1/x^2} = \lim_{x\to 0^+}\frac{\ln\cos x}{x^2} = \lim_{x\to 0^+}\frac{\sec x(-\sin x)}{2x} = -\sec(0)\lim_{x\to 0^+}\frac{\sin x}{2x}.$$

Since the latter expression is of the form $0/0$ we can apply L'Hôpital's rule again to obtain

$$\ln L = -1\cdot\lim_{x\to 0^+}\frac{\cos x}{2} = -\frac{1}{2} \implies L = \frac{1}{\sqrt{e}}.$$

6. The obvious substitution to try is $u=x^{1/2}$, $du=(1/2)x^{-1/2}\,dx$, $2du = dx/\sqrt{x}$ to obtain

$$\int \frac{dx}{\sqrt{x}(1+x)} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u+C = 2\tan^{-1}\sqrt{x}+C.$$

7. We are going to apply the fundamental theorem of calculus, but we need to prepare the expression a little beforehand. Write

$$\int_{\ln x}^{2x} e^{-t^2}\,dt = \int_{\ln x}^{0} e^{-t^2}\,dt + \int_0^{2x} e^{-t^2}\,dt = \int_0^{2x} e^{-t^2}\,dt - \int_0^{\ln x} e^{-t^2}\,dt.$$

Let $F(u)=\int_0^u e^{-t^2}\,dt$. Then the fundamental theorem of calculus says $F'(u)=e^{-u^2}$ so by the chain rule

$$\frac{d}{dx}\int_{\ln x}^{2x} e^{-t^2}\,dt = \frac{d}{dx}F(2x)-\frac{d}{dx}F(\ln x) = F'(2x)\cdot 2 - F'(\ln x)\cdot\frac{1}{x} = 2e^{-(2x)^2}-\frac{1}{x}e^{-(\ln x)^2}.$$