

Math 111 Problem Set 1 Solutions

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1. (a) Let $y = (2x - 1)/(-3x + 2)$ and solve for x :

$$\begin{aligned}(-3x + 2)y &= 2x - 1 \\ 2y + 1 &= 2x + 3xy \\ \frac{2y + 1}{2 + 3y} &= x,\end{aligned}$$

so the inverse function is

$$f^{-1}(x) = \frac{2x + 1}{2 + 3x}.$$

To be totally complete and correct you should give the domain of f^{-1} . You should check your work by verifying that the cancellation equations $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ hold on the appropriate domains.

- (b) Let $y = (x + 2)^5$ and solve for x :

$$\begin{aligned}y^{1/5} &= x + 2 \\ x &= y^{1/5} - 2\end{aligned}$$

so the inverse function is

$$f^{-1}(x) = x^{1/5} - 2.$$

2. (a) $h'(x) = 3x^2 + 2 > 0$ for all x (because $x^2 \geq 0$ for all x and $2 > 0$), so the function h is increasing for all x , so it must be 1-1.
(b) Guessing, $h(1) = 1^3 + 2(1) + 3 = 1 + 2 + 3 = 6$, so $h^{-1}(6) = 1$. By the formula for derivatives of inverse functions,

$$(h^{-1})'(6) = \frac{1}{h'(h^{-1}(6))} = \frac{1}{h'(1)} = \frac{1}{5}.$$

3. (a) By the product rule,

$$\begin{aligned}k'(x) &= \left(\frac{d}{dx}(x^3 + 1)\right)e^x + (x^3 + 1)\frac{d}{dx}e^x \\ &= 3x^2e^x + (x^3 + 1)e^x \\ &= (x^3 + 3x^2 + 1)e^x.\end{aligned}$$

- (b) By the chain rule,

$$\begin{aligned}m'(x) &= \sin'(e^{x^2})\frac{d}{dx}e^{x^2} \\ &= \cos(e^{x^2})e^{x^2}\frac{d}{dx}x^2 \\ &= \cos(e^{x^2})e^{x^2}2x.\end{aligned}$$

4. (a) Let $u = 2x$. Then $du = 2dx$ and

$$\begin{aligned}\int e^{2x} dx &= \int e^u \frac{du}{2} \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{2x} + C.\end{aligned}$$

Check by differentiating.

- (b) Let $u = 2x^2$. Then $du = 4xdx$ and

$$\begin{aligned}\int xe^{2x^2} dx &= \int e^u \frac{du}{4} \\ &= \frac{1}{4} e^u + C \\ &= \frac{1}{4} e^{2x^2} + C.\end{aligned}$$

As usual, check by differentiating.

5. First, find the y -value of the point on the curve. When $x = \pi$, $y = e^x \sin(x) = e^\pi \sin(\pi) = e^\pi \cdot 0 = 0$. So the point on the curve is $(\pi, 0)$. Next find the slope of the tangent line through that point. By the product rule, the derivative of the function is $y' = e^x \sin(x) + e^x \cos(x)$. So the derivative for the given value of x is $y'(x) = y'(\pi) = e^\pi \sin(\pi) + e^\pi \cos(\pi) = e^\pi \cdot 0 + e^\pi \cdot -1 = -e^\pi$. That is the slope m of the tangent line. Since we know a point through which the line passes and the slope of the line, we can write down an equation in point-slope form: $y - 0 = -e^\pi(x - \pi)$. No need to simplify any further.
6. If $y = e^{\lambda x}$ then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$, so $y'' = y' + 6y$ implies $\lambda^2 e^{\lambda x} = \lambda e^{\lambda x} + 6e^{\lambda x}$. Dividing through by $e^{\lambda x}$ (which is safe because $e^{\lambda x}$ is never zero) we obtain the quadratic equation $\lambda^2 = \lambda + 6$ or $\lambda^2 - \lambda - 6 = 0$. Factoring (or using the quadratic solution formula) we obtain $(\lambda - 3)(\lambda + 2) = 0$ so $\lambda = 3$ or $\lambda = -2$ are the only values of λ which satisfy the condition.
7. Divide through by the fastest-increasing function of x , i.e., e^x , and then apply the limit laws to obtain

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2e^x - 1}{-3e^x + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{2e^x}{e^x} - \frac{1}{e^x}}{\frac{-3e^x}{e^x} + \frac{2}{e^x}} \\ &= \lim_{x \rightarrow \infty} \frac{2 - e^{-x}}{-3 + 2e^{-x}} \\ &= \frac{2 - \lim_{x \rightarrow \infty} e^{-x}}{-2 + 2 \lim_{x \rightarrow \infty} e^{-x}}.\end{aligned}$$

We know that $\lim_{x \rightarrow \infty} e^{-x} = 0$ from the textbook, so the answer is $2 / -2 = -1$.

8. The formula for the mass of Carbon 14 we have at any time t in years is $m(t) = m_0 2^{-t/h}$ where m_0 is the initial mass and h is the half-life in years. In this case, $m(t) = 0.32^{-t/(5.73 \times 10^3)}$ (in micrograms). If we know that $m(t) = 0.1$, we have an equation

$$\begin{aligned}0.1 &= 0.32^{-t/(5.73 \times 10^3)} \\ \frac{1}{3} &= 2^{-t/(5.73 \times 10^3)} \\ \log_2(1/3) &= \frac{-t}{5.73 \times 10^3} \\ t &= -5.73 \times 10^3 \log_2(1/3) = 5.73 \times 10^3 \log_2(3).\end{aligned}$$

Since there probably isn't a \log_2 button on your calculator, to get a numerical answer you'll have to use the chain rule $\log_2(3) = \ln(3)/\ln(2)$ to obtain $t = 9.1 \times 10^3$ years.

9. Let's find the first few derivatives and try to find a pattern:

$$\begin{aligned} p^{(1)}(x) &= e^{-x} + (x+50)e^{-x}(-1) = (-x-49)e^{-x} \\ p^{(2)}(x) &= -e^{-x} + (-x-49)e^{-x}(-1) = (x+48)e^{-x} \\ p^{(3)}(x) &= e^{-x} + (x+48)e^{-x}(-1) = (-x-47)e^{-x} \\ p^{(4)}(x) &= -e^{-x} + (-x-47)e^{-x}(-1) = (x+46)e^{-x} \end{aligned}$$

and so on. A good guess at the formula would be $p^{(k)} = (-1)^k(x+50-k)e^{-x}$. (You can prove that formula by induction, but that's not necessary in this course.) So when $k = 100$, $p^{(100)} = (-1)^{100}(x+50-100)e^{-x} = (x-50)e^{-x}$.

10. Let's warm up by proving two simpler results first. We assume that $e^x > 0$ for all x .

- (a) Theorem: $e^x > 1$ for $x > 0$. Proof: Form the function $f(x) = e^x - 1$. Then $f'(x) = e^x > 0$ for all x so f is increasing for $x > 0$. Furthermore $f(0) = e^0 - 1 = 1 - 1 = 0$ so f increases above 0 for $x > 0$, i.e., $f(x) = e^x - 1 > 0$ for $x > 0$, i.e., $e^x > 1$ for $x > 0$.
- (b) Theorem: $e^x > x + 1$ for $x > 0$. Proof: We prove this in a way similar to that of the previous result. Let $g(x) = e^x - (x + 1)$. Then $g'(x) = e^x - 1$, and we know that $e^x - 1 > 0$ for $x > 0$ from the previous result, so $g'(x) > 0$ for $x > 0$ which implies that $g(x)$ is increasing for $x > 0$. Also, $g(0) = e^0 - 0 - 1 = 1 - 1 = 0$, so $g(x)$ is increasing above 0 for $x > 0$, i.e., $g(x) = e^x - (x + 1) > 0$ for $x > 0$, i.e., $e^x > x + 1$ for $x > 0$.
- (c) Now we are ready for the given problem. Theorem: $e^x > x^2/2 + x + 1$ for all $x > 0$. Proof: Let $h(x) = e^x - (x^2/2 + x + 1)$. Then $h'(x) = e^x - x - 1$. By the previous result, $h'(x) > 0$ for $x > 0$ so $h(x)$ is increasing for $x > 0$. Furthermore, $h(0) = e^0 - 0^2/2 - 0 - 1 = 1 - 1 = 0$ so $h(x)$ is increasing above 0 for $x > 0$, i.e., $h(x) = e^x - (x^2/2 + x + 1) > 0$ for $x > 0$, i.e., $e^x > x^2/2 + x + 1$ for $x > 0$.
- (d) Now we use the previous result to estimate the integral. We have

$$1 + x + \frac{x^2}{2} \leq e^x \leq e^1$$

for $0 \leq x \leq 1$. Substituting x^2 for x ,

$$1 + x^2 + \frac{x^4}{2} \leq e^{x^2} \leq e^1$$

for $0 \leq x^2 \leq 1$, i.e., for $0 \leq x \leq 1$. (In fact, the result holds for $-1 \leq x \leq 0$ too; why?)

Integrating throughout the previous inequality we have

$$\int_0^1 1 + x^2 + \frac{x^4}{2} dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^1 dx,$$

i.e.,

$$x + \frac{x^3}{3} + \frac{x^5}{10} \Big|_0^1 \leq \int_0^1 e^{x^2} dx \leq e^1 x \Big|_0^1.$$

Evaluating the integrals, we have

$$1 + \frac{1}{3} + \frac{1}{10} \leq \int_0^1 e^{x^2} dx \leq e$$

which shows that the given integral lies between 1.43 and 2.71, with an error of 1.28 or so.