Math 111 Problem Set 4 Solutions

Edward Doolittle

October 27, 2005

1. (a) Let $f(x) = x$, $g'(x) = \sin 7x$. Then $f'(x) = 1$, $g(x) = -(1/7)\cos 7x$, and the integration by parts formula gives

$$\int x \sin 7x \, dx = -\frac{1}{7} x \cos 7x + \frac{1}{7} \int \cos 7x \, dx = -\frac{1}{7} x \cos 7x + \frac{1}{49} \sin 7x + C.$$

You should check the answer by differentiating.

(b) Similar to the previous answer, let $f(r) = r$, $g'(r) = e^{\pi r}$. Then $f'(r) = 1$, $g(r) = (1/\pi)e^{\pi r}$, and the integration by parts formula gives

$$\int re^{\pi r} \, dr = \frac{1}{\pi} re^{\pi r} - \frac{1}{\pi} \int e^{\pi r} \, dr = \frac{1}{\pi} re^{\pi r} - \frac{1}{\pi^2} e^{\pi r} + C.$$

Again, you should check the answer by differentiating.

(c) Following the usual trick for evaluating the integrals of inverse trig functions, introduce a factor of 1 and then apply integration by parts with $f(x) = \cos^{-1} x$, $g'(x) = 1$, $f'(x) = -1/\sqrt{1-x^2}$, $g(x) = x$:

$$\int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx.$$

Now, making the substitution $u = 1 - x^2$, $du = -2x \, dx$, $x \, dx = du/(-2)$,

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \frac{1}{2} \int \frac{du}{u^{1/2}} = x \cos^{-1} x - \frac{1}{2} \frac{u^{1/2}}{1/2} + C = x \cos^{-1} x - (1-x^2)^{1/2} + C.$$

As usual, check by differentiating.

(d) This is a case where integration by parts doesn't seem to go anywhere; we need to aply it twice and solve for the required integral. First, let $f(\theta) = \sin(2\theta)$, $g'(\theta) = e^{3\theta}$. Then $f'(\theta) = 2\cos(2\theta)$, $g(\theta) = (1/3)e^{3\theta}$, and

$$\int e^{3\theta} \sin(2\theta) \, d\theta = \frac{1}{3} e^{3\theta} \sin(2\theta) - \frac{2}{3} \int e^{3\theta} \cos(2\theta) \, d\theta. \tag{1}$$

Let's try again, this time with $f(\theta) = \cos(2\theta)$, $g'(\theta) = e^{3\theta}$, $f'(\theta) = -2\sin(2\theta)$, $g(\theta) = (1/3)e^{3\theta}$:

$$\int e^{3\theta} \cos(2\theta) \, d\theta = \frac{1}{3} e^{3\theta} \cos(2\theta) + \frac{2}{3} \int e^{3\theta} \sin(2\theta) \, d\theta. \tag{2}$$

Substituting (2) into (1),

$$\int e^{3\theta} \sin(2\theta) \, d\theta = \frac{1}{3} e^{3\theta} \sin(2\theta) - \frac{2}{9} e^{3\theta} \cos(2\theta) - \frac{4}{9} \int e^{3\theta} \sin(2\theta) \, d\theta.$$

Solving for the required integral,

$$\int e^{3\theta} \sin(2\theta) \, d\theta = \frac{9}{13} \left(\frac{1}{3} e^{3\theta} \sin(2\theta) - \frac{2}{9} e^{3\theta} \cos(2\theta) \right) + C.$$

As usual, you should check the result by differentiation.

2. (a) The power of $\cos x$ is odd, so convert as many $\cos x$ to $\sin x$ as possible and make the substitution $u = \sin x$:

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x(1-\sin^2 x)\cos x\, dx = \int (u^2-u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$$

Check by differentiating and applying trig identities.

(b) The power of $\sin x$ is odd, so convert as many $\sin x$ to $\cos x$ as possible and make the substitution $u = \cos x$, $-du = \sin x\, dx$:

$$\int \sin^5 x \cos^4 x\, dx = \int (1-\cos^2 x)^2 \cos^4 x \sin x\, dx = -\int (1-u^2)^2 u^4\, du.$$

Expanding the polynomial and integrating,

$$\int \sin^5 x \cos^4 x\, dx = -\int (1-2u^2+u^4)u^4\, du = -\int (u^4 - 2u^6 + u^8)\, du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C.$$

Reversing the substitution,

$$\int \sin^5 x \cos^4 x\, dx = -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C.$$

Check by differentiating and applying trig identities.

(c) Let I be the integral in question. The powers of $\sin x$ and $\cos x$ are both even, so pair off sines and cosines and use double angle identities:

$$I = \int \sin^2\theta(\sin\theta\cos\theta)^2\, d\theta = \int \frac{1-\cos(2\theta)}{2}\frac{\sin^2(2\theta)}{4}\, d\theta. = \frac{1}{8}\int (\sin^2(2\theta) - \cos(2\theta)\sin^2(2\theta))\, d\theta.$$

The first term can be evaluated by another double angle formula:

$$\frac{1}{8}\int \sin^2(2\theta)\, d\theta = \frac{1}{16}\int 1-\cos(4\theta)\, d\theta = \frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) + C.$$

The second term can be evaluated by the substitution $u = \sin(2\theta)$, $du = 2\cos(2\theta)\, d\theta$, $du/2 = \cos(2\theta)\, d\theta$:

$$-\frac{1}{8}\int \cos(2\theta)\sin^2(2\theta)\, d\theta = -\frac{1}{16}\int u^2\, du = -\frac{1}{48}u^3 + C = -\frac{1}{48}\sin^3(2\theta) + C.$$

Assembling the above results,

$$\int \sin^4\theta\cos^2\theta\, d\theta = \frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) - \frac{1}{48}\sin^3(2\theta) + C.$$

As usual, you should check by differentiating and applying trig identities.

(d) The power of $\sec t$ is even so convert $\sec^2$ to tan (leaving $\sec^2$ in the integrand) and make the substitution $u = \tan t$, $du = \sec^2 t\, dt$:

$$\int \tan^2 t \sec^4 t\, dt = \int \tan^2 t(1+\tan^2 t)\sec^2 t\, dt = \int u^2(1+u^2)\, du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{1}{3}\tan^3 t + \frac{1}{5}\tan^5 t + C.$$

As usual, check by differentiating and applying trig identities.

3. It's probably best to evaluate the indefinite integrals first, because then you can check your results by differentiating. However, just for the sake of variety and for the sake of applying a technique that we have learned that may be useful in some circumstances, I have solved the problems in this section using the integration by parts rule for definite integrals. You should check these results by evaluating the indefinite integrals first and then evaluating the definite integrals.

(a) Let $f(t)=t$, $g'(t)=\cos 4t$. Then $f'(t)=1$ and $g(t)=(1/4)\sin 4t$ and

$$\int_0^\pi t\cos 4t\,dt = \frac{1}{4}t\sin 4t\Big|_0^\pi - \frac{1}{4}\int_0^\pi \sin 4t\,dt = 0+\frac{1}{16}\cos 4t\Big|_0^\pi = 0.$$

(b) Let $f(x)=\ln x$, $g'(t)=x^{-3}$; then $f'(x)=1/x$, $g(x)=x^{-2}/(-2)$, and the integration by parts formula gives

$$\int_1^2 \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2}\Big|_1^2 + \frac{1}{2}\int_1^2 x^{-3}\,dx = -\frac{1}{8}\ln 2 - \frac{1}{4}x^{-2}\Big|_1^2 = -\frac{1}{8}\ln 2 - \frac{1}{16}+\frac{1}{4}.$$

(c) Let $f(x)=x^2+4$, $g'(x)=e^x$; then $f'(x)=2x$, $g(x)=e^x$, and

$$\int_1^4 (x^2+4)e^x\,dx = (x^2+4)e^x\Big|_1^4 - \int_1^4 2xe^x\,dx = 20e^4-5e-2\int_1^4 xe^x\,dx. \tag{3}$$

We need to integrate by parts again. Letting $f(x)=x$, $g'(x)=e^x$, we have $f'(x)=1$, $g(x)=e^x$, and

$$\int_1^4 xe^x\,dx = xe^x\Big|_1^4 - \int_1^4 e^x\,dx = 4e^4-e-e^4+e=3e^4. \tag{4}$$

Substituting (4) into (3) gives

$$\int_1^4 (x^2+4)e^x\,dx = 20e^4-5e-2(3e^4)=14e^4-5e.$$

(d) Let $f(t)=t$, $g'(t)=2^t$; then $f'(t)=1$, $g(t)=(1/\ln 2)2^t$, and

$$\int_0^8 t2^t\,dt = \frac{1}{\ln 2}t2^t\Big|_0^8 - \frac{1}{\ln 2}\int_0^8 2^t\,dt = \frac{2^{11}}{\ln 2} - \frac{1}{(\ln 2)^2}2^t\Big|_0^8 = \frac{2^{11}}{\ln 2} - \frac{2^8}{(\ln 2)^2}+\frac{1}{(\ln 2)^2}.$$

4. (a) First evaluate the indefinite integral. The obvious thing to try is the substitution $u=\sqrt{x}$, $du=dx/(2\sqrt{x})$, $2u\,du=dx$. After the substitution we integrate by parts:

$$\int \cos\sqrt{x}\,dx = \int 2u\cos u\,du = -2u\sin u+2\cos u+C = 2\cos\sqrt{x}-2\sqrt{x}\sin\sqrt{x}+C.$$

(Check by differentiating.) Now, evaluating the definite integral,

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \cos\sqrt{x}\,dt = 2\cos(\pi^{1/4})-2\pi^{1/4}\sin(\pi^{1/4})-2\cos((\pi/2)^{1/4})+2(\pi/2)^{1/4}\sin((\pi/2)^{1/4}).$$

(b) As above, let $u=\sqrt{x}$, $2u\,du=dx$ and integrate by parts:

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du = 2ue^u-2e^u+C = 2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C.$$

(Check by differentiating.) Evaluating the definite integral,

$$\int_4^9 e^{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}\Big|_4^9 = 6e^3-2e^3-4e^2+2e^2=4e^3-2e^2.$$

(c) Let I be the desired integral. The power of tan is odd so change as many tan to sec as possible and substitute $u=\sec\theta$:

$$I=\int(\sec^2\theta-1)\sec^2\theta\sec\theta\tan\theta\,d\theta = \int u^4-u^2\,du = \frac{1}{5}u^5-\frac{1}{3}u^3+C = \frac{1}{5}\sec^5\theta-\frac{1}{3}\sec^3\theta+C.$$

As usual, check by differentiating and applying trig identities.

(d) The function $\cot x$ behaves like $\tan x$ in trig identities and differentiation formulas. The strategy is to change $\cot^2$ into terms of $\csc^2$ using the Pythagorean identity $\cot^2 x + 1 = \csc^2 x$:

$$\int \cot^4 x\,dx = \int \cot^2 x \cot^2 x\,dx = \int \cot^2 x \csc^2 x - \cot^2 x\,dx. \tag{5}$$

The first integral in the right side of (5) can be evaluated by the substitution $u = \cot x$, $-du = \csc^2 x\,dx$:

$$\int \cot^2 x \csc^2 x\,dx = -\int u^2\,du = -\frac{u^3}{3} + C = -\frac{1}{3}\cot^3 x + C.$$

The second integral in the right side of (5) can be evaluated by again applying the Pythagorean identity:

$$-\int \cos^2 x\,dx = \int -\csc^2 x + 1\,dx = \cot x + x + C.$$

Assembling our results,

$$\int \cot^4 x\,dx = -\frac{1}{3}\cot^3 x + \cot x + x + C.$$

(Check by differentiation and application of trig identities.) Evaluating the definite integral,

$$\int_{\pi/4}^{\pi/2} \cot^4 x\,dx = \left. -\frac{1}{3}\cot^3 x + \cot x + x \right|_{\pi/4}^{\pi/2} = \frac{\pi}{2} + \frac{1}{3} - 1 - \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}.$$

5. (a) Multiply the integrand by the 'conjugate trig function' $(1+\sin x)/(1+\sin x)$:

$$\int \frac{dx}{1-\sin x} = \int \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx = \int \frac{1+\sin x}{\cos^2 x}\,dx.$$

The integral can now be written in terms of $\sec x$ and $\tan x$:

$$\int \frac{dx}{1-\sin x} = \int \sec^2 x + \sec x \tan x\,dx = \tan x + \sec x + C.$$

Alternatively, you can write the answer as

$$\int \frac{dx}{1-\sin x} = \frac{1+\sin x}{\cos x} + C.$$

Check by differentiating.

(b) The power of $\tan\theta$ is even, and the power of $\sec\theta$ is odd, so there is no simple substitution. Integration by parts is necessary in this case, but you can reduce the amount of work required by changing all the $\tan\theta$ to $\sec\theta$,

$$\int \tan^4\theta \sec\theta\,d\theta = \int (\sec^2\theta - 1)^2 \sec\theta\,d\theta = \int \sec^5\theta - 2\sec^3\theta + \sec\theta\,d\theta,$$

and applying a known reduction formula, namely formula 77 from the integral tables at the back of the textbook:

$$\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du \tag{6}$$

Recall from the lectures that

$$\int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C. \tag{7}$$

By reduction formula (6) and formula (7),

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\tan\theta + \frac{1}{2}\ln|\tan\theta + \sec\theta| + C. \tag{8}$$

By reduction formula (6) and formula (8),

$$\int \sec^5\theta\, d\theta = \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{4}\int \sec^3\theta\, d\theta = \frac{1}{4}\tan\theta\sec^3\theta + \frac{3}{8}\tan\theta + \frac{3}{8}\ln|\tan\theta + \sec\theta| + C.$$

Assembling the above results,

$$\int \tan^4\theta\sec\theta\, d\theta = \frac{1}{4}\tan\theta\sec^3\theta - \frac{5}{8}\tan\theta + \ln|\tan\theta + \sec\theta| + C.$$

Evaluating the definite integral,

$$\int_0^{\pi/4} \tan^4\theta\sec\theta\, d\theta = \frac{1}{4}(1)(\sqrt{2})^3 - \frac{5}{8}(1) + \ln(1 + 2^{1/2}) = \frac{1}{\sqrt{2}} - \frac{5}{8} + \ln(1 + \sqrt{2}).$$

6. We're going to end up in some difficulty with notation further down the line, so I recommend changing f to g in the statement of the problem; now we need to prove that

$$\int_{g(a)}^{g(b)} g^{-1}(y)\ dy = bg(b) - ag(a) + \int_a^b g(x)\ dx. \tag{9}$$

This is one case where it's very helpful to use the definite integral forms of the substitution rule

$$\int_{g(a)}^{g(b)} f(y)\ dy = \int_a^b f(g(x))\ g'(x)\ dx \tag{10}$$

and the integration by parts rule

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx. \tag{11}$$

Let's work on the left hand side of (9). First use the substitution rule (10) with $f = g^{-1}$, followed by the cancellation equation $g^{-1}(g(x)) = x$:

$$\int_{g(a)}^{g(b)} g^{-1}(y)\ dy = \int_a^b g^{-1}(g(x))g'(x)\ dx = \int_a^b xg'(x)\ dx$$

Now applying the integration by parts rule (11) with $f(x) = x$, $f'(x) = 1$,

$$\int_a^b xg'(x)\ dx = xg(x)\Big|_a^b - \int_a^b g(x)\ dx.$$

Putting it all together,

$$\int_{g(a)}^{g(b)} g^{-1}(y)\ dy = bg(b) - ag(a) - \int_a^b g(x)\ dx,$$

as required. Without the definite integral forms of the substitution and integration by parts rules, the problem is rather messy.

The result is illustrated in Figures 1 and 2, which also provide an alternate argument for the result. Figure 2 is just Figure 1 flipped along the line $y = x$. We calculate the areas of various regions in the diagrams. Region A is just a rectangle, so its area is $af(a)$. Region B is the region under the curve $y = f(x)$, so its area is $\int_a^b f(x)\ dx$. Region C in Figure 2 is just the region under the curve $x = f^{-1}(y)$ so its area is $\int_{f(a)}^{f(b)} f^{-1}(y)\ dy$. The sum of the three areas A, B, and C is a rectangle the area of which is $bf(b)$. Therefore

$$af(a) + \int_a^b f(x)\ dx + \int_{f(a)}^{f(b)} f^{-1}(y)\ dy = bf(b)$$

which is equivalent to the required result. (Question: what happens if $f(x)$ is negative for some values of x? What happens if a is negative?)

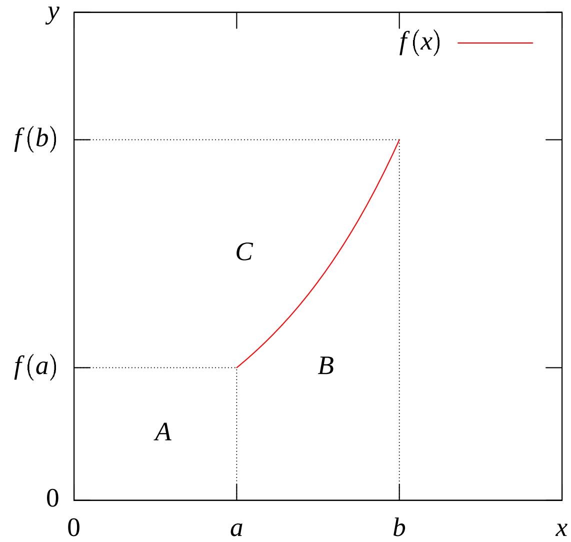

Figure 1: Graph of $f(x)$, $a \leq x \leq b$

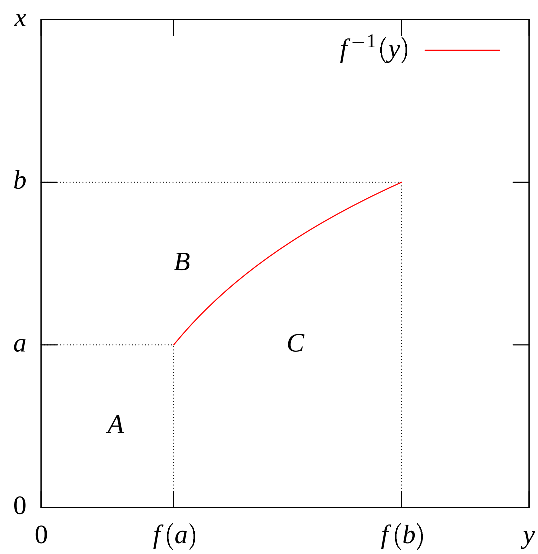

Figure 2: Graph of $f^{-1}(y)$, $f(a) \leq y \leq f(b)$