Math 111 Problem Set 5 Solutions

Edward Doolittle

November 21, 2005

1. (a) Make the substitution $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\, d\theta$:

$$\int \frac{\sqrt{x^2-16}}{x^4}\, dx = \int \frac{\sqrt{16\sec^2\theta - 16}}{4^4\sec^4\theta} 4\sec\theta\tan\theta\, d\theta = \int \frac{4\tan\theta}{4^4\sec^3\theta}\tan\theta\, d\theta = \frac{1}{64}\int \sin^2\theta\cos\theta\, d\theta.$$

Now making the transformation $u = \sin\theta$, $du = \cos\theta\, d\theta$,

$$\frac{1}{64}\int \sin^2\theta\cos\theta\, d\theta = \frac{1}{64}\int u^2\, du = \frac{1}{64}\frac{u^3}{3} + C = \frac{1}{192}\sin^3\theta + C.$$

By the right triangle with sides $(A,O,H) = (4, \sqrt{x^2-16}, x)$, $\sin\theta = \sqrt{x^2-16}/x$, so putting it all together

$$\int \frac{\sqrt{x^2-16}}{x^4}\, dx = \frac{(x^2-16)^{3/2}}{192x^3} + C.$$

Check by differentiating.

(b) It's probably easier to make an algebraic substition here; try it. However, the trig substitution also works. Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$. Then

$$\int \frac{x}{\sqrt{9-x^2}}\, dx = \int \frac{3\sin\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\theta\, d\theta = \int 3\sin\theta\, d\theta = -3\cos\theta + C.$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$ or the appropriate right triangle,

$$\int \frac{x}{\sqrt{9-x^2}}\, dx = -3\sqrt{1-\sin^2\theta} + C = -3\sqrt{1-(x/3)^2} + C = -\sqrt{9-x^2} + C.$$

Check by differentiating. The definite integral is now

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{9-x^2}}\, dx = \sqrt{9} - \sqrt{9-5} = 3 - 2 = 1.$$

(c) Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$. Then

$$\int \frac{x^2}{(a^2-x^2)^{5/2}}\, dx = \int \frac{a^2\sin^2\theta}{a^5\cos^5\theta} a\cos\theta\, d\theta = \frac{1}{a^2}\int \tan^2\theta\sec^2\theta\, d\theta.$$

Now making the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$,

$$\frac{1}{a^2}\int \tan^2\theta\sec^2\theta\, d\theta = \frac{1}{a^2}\int u^2\, du = \frac{1}{a^2}\frac{u^3}{3} + C = \frac{\tan^3\theta}{3a^2} + C.$$

By using a right triangle with the appropriate sides,

$$\int \frac{x^2}{(a^2-x^2)^{5/2}}\, dx = \frac{1}{3a^2}\frac{x^3}{(a^2-x^2)^{3/2}} + C.$$

Check by differentiating.

(d) Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$. Then

$$\int \frac{x^2}{(a^2+x^2)^{5/2}}\,dx = \int \frac{a^2\tan^2\theta}{a^5\sec^5\theta} a\sec^2\theta\, d\theta = \frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta.$$

Since the power of cos is odd, make the substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta = \frac{1}{a^2}\int u^2\, du = \frac{1}{a^2}\frac{u^3}{3} + C$$

Reversing the substitutions using the appropriate right triangle,

$$\int \frac{x^2}{(a^2+x^2)^{5/2}}\,dx = \frac{1}{a^2}\frac{\sin^3\theta}{3} + C = \frac{1}{3a^2}\frac{x^3}{(a^2+x^2)^{3/2}} + C.$$

Check by differentiating.

2. (a) Let $4x = 3\sec\theta$, $4dx = 3\sec\theta\tan\theta\, d\theta$. Then

$$\int \frac{dx}{\sqrt{16x^2-9}} = \frac{3}{4}\int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{4}\int \sec\theta d\theta = \frac{1}{4}\ln|\sec\theta+\tan\theta| + C.$$

Reversing the substitution, $\tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{16x^2-9}/3$ and

$$\int \frac{dx}{\sqrt{16x^2-9}} = \frac{1}{4}\ln|4x/3 - \sqrt{16x^2-9}/3| + C = \frac{1}{4}\ln|4x - \sqrt{16x^2-9}| + C$$

where $-(1/4)\ln 3$ has been absorbed into C. Check by differentiating.

(b) Here the algebraic substitution may work better than a trig substitution. Let $u = 4-9x^2$, $du = -18x\, dx$. Then

$$\int x\sqrt{4-9x^2}\,dx = \int u^{1/2}\frac{du}{-18} = -\frac{1}{18}\frac{u^{3/2}}{3/2} + C = -\frac{1}{27}(4-9x^2)^{3/2} + C.$$

(Check by differentiating.) Evaluating the definite integral,

$$\int_0^{2/3} x\sqrt{4-9x^2}\,dx = \frac{1}{27}0\sqrt{4-9(0)^2} - \frac{1}{27}\frac{2}{3}\left(4-9\left(\frac{2}{3}\right)^2\right) = 0-0 = 0.$$

(c) Make the trig substitution $2x = \tan\theta$, $2dx = \sec^2\theta\, d\theta$, we have

$$\int \sqrt{4x^2+1}\,dx = \int \sqrt{\sec^2\theta}\sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta.$$

By example 8 on page 523 of the textbook,

$$\int \sqrt{4x^2+1}\,dx = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C = x\sqrt{4x^2+1} + \frac{1}{2}\ln|\sqrt{4x^2+1} + 2x| + C.$$

Check by differentiating. Now the definite integral is

$$\int_0^1 \sqrt{4x^2+1}\,dx = 1\sqrt{5} + \frac{1}{2}\ln|\sqrt{5}+2| - 0 - \frac{1}{2}\ln|\sqrt{1}+0| = \sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5}).$$

(d) Let $ax = b\sec\theta$, $a\,dx = b\sec\theta\tan\theta\, d\theta$. Then

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = \frac{b}{a}\int \frac{\sec\theta\tan\theta\, d\theta}{b^5\tan^5\theta} = \frac{1}{ab^4}\int \frac{\cos^3\theta}{\sin^4\theta}\, d\theta = \frac{1}{ab^4}\int \cot^3\theta\csc\theta\, d\theta.$$

Since the power of cot is odd we should convert as many cot to csc as possible and make the substitution $u = \csc\theta$, $du = -\cot\theta\csc\theta\, d\theta$:

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = \frac{1}{ab^4}\int(\csc^2\theta - 1)\cot\theta\csc\theta\, d\theta = -\frac{1}{ab^4}\int(u^2-1)\, du = -\frac{1}{ab^4}\left(\frac{u^3}{3} - u\right) + C.$$

Reversing the substitutions using a $(A, O, H) = (b, (a^2x^2 - b^2)^{1/2}, ax)$ right triangle,

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = -\frac{1}{ab^4}\left(\frac{\csc^3\theta}{3} - \csc\theta\right) + C = -\frac{1}{ab^4}\left(\frac{a^3x^3}{3(a^2x^2-b^2)^{3/2}} - \frac{ax}{(a^2x^2-b^2)^{1/2}}\right) + C.$$

Check by differentiating.

3. These problems require completing the square.

(a) Write $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x+1)^2 + 1$. Then making the substitution $x + 1 = \tan\theta$, $dx = \sec^2\theta\, d\theta$,

$$\int \frac{dx}{(x^2+2x+2)^4} = \int \frac{\sec^2\theta\, d\theta}{\sec^8\theta} = \int \cos^6\theta\, d\theta.$$

By the hint,

$$\int \frac{dx}{(x^2+2x+2)^4} = \frac{1}{6}\sin\theta\cos^5\theta + \frac{5}{24}\sin\theta\cos^3\theta + \frac{15}{48}\sin\theta\cos\theta + \frac{15}{48}\theta + C.$$

Reversing the substitution by a $(1, x+1, \sqrt{x^2+2x+2})$ right triangle, $\sin\theta = (x+1)/\sqrt{x^2+2x+2}$, $\cos\theta = 1/\sqrt{x^2+2x+2}$, and

$$\int \frac{dx}{(x^2+2x+2)^4} = \frac{1}{6}\frac{x+1}{(x^2+2x+2)^3} + \frac{5}{24}\frac{x+1}{(x^2+2x+2)^2} + \frac{15}{48}\frac{x+1}{x^2+2x+2} + \frac{15}{48}\tan^{-1}(x+1) + C.$$

(b) Completing the square, $t^2 - 6t + 1 = t^2 - 6t + 9 - 8 = (t-3)^2 - 8$. Making the substitution $t - 3 = 2\sqrt{2}\sec\theta$, $dt = 2\sqrt{2}\sec\theta\tan\theta\, d\theta$,

$$\int \frac{3\, dt}{\sqrt{t^2-6t+1}} = \int \frac{6\sqrt{2}\sec\theta\tan\theta\, d\theta}{2\sqrt{2}\tan\theta} = \int 3\sec\theta\, d\theta = 3\ln|\sec\theta + \tan\theta| + C.$$

Reversing the substitution, $\sec\theta = (t-3)/(2\sqrt{2})$, $\tan\theta = \sqrt{(t-3)^2/8 - 1} = \sqrt{t^2-6t+1}/(2\sqrt{2})$,

$$\int \frac{3\, dt}{\sqrt{t^2-6t+1}} = 3\ln\left|\frac{t-3}{2\sqrt{2}} + \frac{\sqrt{t^2-6t+1}}{2\sqrt{2}}\right| + C = 3\ln|t - 3 + \sqrt{t^2-6t+1}| + C$$

by the law of logarithms $\ln|a/b| = \ln a - \ln b$, absorbing the constant $-\ln 2\sqrt{2}$ into C. (Check by differentiating.)

(c) As with the previous problem, complete the square to obtain $t^2 - 6t + 1 = (t-3)^2 - 8$, but this time make the algebraic substitution $u = (t-3)^2 - 8$, $du = 2(t-3)\, dt$:

$$\int \frac{t-3}{\sqrt{t^2-6t+1}}\, dt = \int \frac{du}{2u^{1/2}} = \frac{1}{2}\frac{u^{1/2}}{1/2} + C = \sqrt{t^2-6t+1} + C.$$

Check by differentiating.

(d) Adding the previous two results,

$$\int \frac{t}{\sqrt{t^2-6t+1}}\, dt = \int \frac{t-3}{\sqrt{t^2-6t+1}}\, dt + \int \frac{3}{\sqrt{t^2-6t+1}}\, dt = \sqrt{t^2-6t+1} + 3\ln|t - 3 + \sqrt{t^2-6t+1}| + C.$$

As usual, check by differentiating.

4. (a) For large x, the integrand is approximately $x/x^2 = 1/x$, and we know that $\int_a^\infty (1/x)\,dx$ is divergent, so the given integral is likely divergent. To make sure we use the comparison test, bounding the integrand from below by some multiple of $1/x$:

$$1 + x + x^2 < 2x^2$$

for x large enough (specifically for x bigger than the largest root of the quadratic equation $1 + x + x^2 = 2x^2$), so

$$\frac{1}{2x} < \frac{x}{1 + x + x^2}$$

for x large enough. However,

$$\frac{1}{2}\int_1^\infty \frac{1}{x}\,dx$$

diverges, so it follows that

$$\int_1^\infty \frac{x}{1 + x + x^2}\,dx$$

diverges as well by the comparison test. Changing the lower bound of integration from 1 to 0 doesn't make any difference in the divergence of an improper integral, so the given integral diverges.

Alternatively, the integral may be evaluated explicitly using techniques similar to those of 3(d) above.

(b) The integrand is approximately $1/t^2$ for large t so we should expect it to converge. However, we also need to check whether there is any type II "improperness" since the denominator of the integrand may vanish at some point. Factoring the denominator, $t^2 + 4t + 3 = (t+1)(t+3)$, so the denominator vanishes at $t = -1$ and $t = -3$; those points are outside of the interval over which we are integrating, namely $[0, \infty)$, so we don't need to worry about the integrand going infinte in the domain of integration.

Since $t^2 < t^2 + 4t + 3$ for $t > 0$,

$$\frac{1}{t^2 + 4t + 3} < \frac{1}{t^2}$$

for t large enough; since $\int_1^\infty (1/t^2)\,dt$ converges, the given integral converges by the comparison test.

In order to evaluate the integral, complete the square in the denominator and make the substitution $t + 2 = \sec\theta$, $dt = \sec\theta\tan\theta\,d\theta$ to obtain

$$\int \frac{1}{t^2 + 4t + 3}\,dt = \int \frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta = \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C$$

by the hint. Reversing the substitution by the $(A, O, H) = (1, \sqrt{t^2 + 4t + 3}, t + 2)$ right triangle,

$$\int \frac{1}{t^2 + 4t + 3}\,dt = \ln\left|\frac{t+2}{\sqrt{t^2 + 4t + 3}} - \frac{1}{\sqrt{t^2 + 4t + 3}}\right| + C = \ln|t+1| - \frac{1}{2}\ln|t^2 + 4t + 3| + C.$$

Check by differentiating. To evaluate the definite improper integral,

$$\int_0^\infty \frac{dt}{t^2 + 4t + 3} = \lim_{M\to\infty}\int_0^M \frac{dt}{t^2 + 4t + 3} = \lim_{M\to\infty} \ln\left|\frac{M+1}{\sqrt{M^2 + 4M + 3}}\right| - \ln\left|\frac{0+1}{0+0+3}\right| = \ln 3$$

where the limit was obtained by dividing through by M in the fraction inside the logarithm.

Since the limit exists, the comparison test wasn't really necessary, but in general it's a good idea to quickly check the integral using the comparison test before putting too much of an effort into evaluating it.

(c) The integrand blows up as x approaches 2 from below, so the integral is improper of type II. In this particular case it's probably easier to evaluate the integral, but further down I'll explain how to use the comparison test to check for convergence in this case. To evaluate the integral, let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$ to obtain

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \theta + C = \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Therefore

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{a\to 2^-} \int_0^a \frac{dx}{\sqrt{4-x^2}} = \lim_{a\to 2^-} \sin^{-1}\left(\frac{a}{2}\right) - \sin^{-1}(0) = \frac{\pi}{2}$$

and the integral is convergent to the value $\pi/2$.

It is possible to check for convergence using the comparison test, but it is a little more difficult than in the other cases we've studied. The idea is to factor out the part that goes to infinity from the integrand and show that the rest is approximately constant:

$$\frac{1}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2+x}}\frac{1}{\sqrt{2-x}}.$$

Near the singular point $x = 2$ the first factor is approximately $1/\sqrt{2+x} = 1/\sqrt{4} = 1/2$, while the second factor is of the order $u^{-1/2}$ where $u = 2-x$. Let's make that change of variables so that the improper integral becomes

$$\int_0^2 (4-u)^{-1/2} u^{-1/2}\, du < \int_0^2 u^{-1/2}\, du$$

because $(4-u)^{-1/2} < 1$ on the interval $[0,2]$. The latter integral is convergent by the "p-test" because the power of u is greater than -1; by the comparison test the given integral is convergent. This still leaves us with the problem of evaluating the integral, which we then take care of as in the first part of this solutions.

(d) In this case it isn't hard to check using the comparison test. The integrand blows up near $z = 2$, so the integral is improper of type II at $z = 2$; the integral really means

$$\int_0^3 (z-2)^{-1/3}\, dz = \lim_{a\to 2^-} \int_0^a (z-2)^{-1/3}\, dz + \lim_{b\to 2^+} \int_b^3 (z-2)^{-1/3}\, dz. \qquad (1)$$

For the latter integral we let $u = z-2$ which leads to comparison with $\int_0^1 u^{-1/3}$ which is convergent by the p-test. Similar reasoning applies to $\int_0^a (z-2)^{-1/3}\, dz$ with the change of variables $v = 2-z$.

Now that we know that the integral is convergent, we can evaluate it as if it were not improper. Making the change of variables $u = z-2$,

$$\int (z-2)^{-1/3}\, dz = \int u^{-1/3}\, du = \frac{u^{2/3}}{2/3} + C = \frac{3}{2}(z-2)^{2/3} + C$$

so the definite integral is

$$\int_0^3 (z-2)^{-1/3}\, dz = \frac{3}{2}(3-2)^{2/3} - \frac{3}{2}(0-2)^{2/3} = \frac{3}{2}(1) - \frac{3}{2}\sqrt[3]{4}.$$

Alternatively, we could have evaluated the limits in (1) to obtain the same result.

5. Taking the limit throughout the given formula,

$$\lim_{t\to\infty} \int_0^t x^n e^{ax}\, dx = \lim_{t\to\infty} \frac{1}{a} t^n e^{at} - \frac{1}{a} 0^n e^{a\cdot 0} - \frac{n}{a} \lim_{t\to\infty} \int_0^t x^{n-1} e^{ax}\, dx;$$

the second limit above is 0 by the hint so the formula becomes

$$\int_0^\infty x^n e^{ax}\,dx = -\frac{n}{a}\int_0^\infty x^{n-1}e^{ax}\,dx$$

which is actually simpler than the corresponding formula on a finite interval. It follows that

$$\int_0^\infty x^2 e^{-3x}\,dx = \frac{2}{3}\int_0^\infty x^1 e^{-3x}\,dx = \frac{2}{3}\frac{1}{3}\int_0^\infty x^0 e^{-3x}\,dx = \frac{2}{9}\,\frac{1}{-3}e^{-3x}\Big|_0^\infty = \frac{2}{27}.$$

(Question: this improper integral can give us a generalization of the factorial function to any nonnegative real number; how?)

6. Make the substitution $u = \cos t$, $du = -\sin t\,dt$ to obtain

$$\int \frac{\sin t\,dt}{\sqrt{1+\cos^2 t}} = \int \frac{-du}{\sqrt{1+u^2}}.$$

At this point a second trig substitution is called for. Let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, to obtain

$$\int \frac{-du}{\sqrt{1+u^2}} = \int \frac{-\sec^2\theta\,d\theta}{\sec\theta} = -\int \sec\theta\,d\theta = -\ln|\sec\theta+\tan\theta| + C.$$

Reversing the substitutions,

$$\int \frac{\sin t\,dt}{\sqrt{1+\cos^2 t}} = -\ln|u+\sqrt{1+u^2}| + C = -\ln|\cos t + \sqrt{1+\cos^2 t}| + C.$$

Check by differentiating. To evaluate the definite integral,

$$\int_0^{\pi/2} \frac{\sin t\,dt}{\sqrt{1+\cos^2 t}} = -\ln|0+\sqrt{1+0}| + \ln|1+\sqrt{1+1}| = \ln(1+\sqrt{2}).$$