

Math 111 Problem Set 7 Solutions

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1. You can start counting at 0 (if it makes sense) or at 1, either is fine.

$$(a) a_0 = \frac{1}{1}, a_1 = -\frac{5}{1}, a_2 = \frac{25}{2}, a_3 = -\frac{125}{6}, a_4 = \frac{625}{24}, a_5 = -\frac{3125}{120}$$

$$(b) b_0 = 2, b_1 = 2(1-x), b_2 = 2(1-x)^2, b_3 = 2(1-x)^3, b_4 = 2(1-x)^4, b_5 = 2(1-x)^5$$

2. There are an infinite number of formulas that answer each question. I have chosen two which I feel are the simplest. Here we will start numbering terms of the sequences at $n = 1$.

$$(a) a_n = 4n - 1$$

$$(b) a_n = (-1)^n \frac{n^2}{2^n}$$

3. (a) The terms can be re-written as geometric with $r = 12$ so the sequence diverges to ∞ : $a_n = \frac{1}{4} \cdot \frac{3^n}{4^{-n}} = \frac{1}{4} \cdot 12^n$.

(b) By L'Hôpital's rule,

$$\lim_{x \rightarrow \infty} \frac{x^e}{e^x} = \lim_{x \rightarrow \infty} \frac{ex^{e-1}}{e^x} = \lim_{x \rightarrow \infty} \frac{e(e-1)x^{e-2}}{e^x} = \lim_{x \rightarrow \infty} \frac{e(e-1)(e-2)x^{e-3}}{e^x}$$

where each of the limits except the last is of the form ∞/∞ (which you must check otherwise you cannot apply L'Hôpital's rule). Since $e < 3$, the last limit tends to $0/\infty = 0$, and therefore the sequence converges to 0.

4. (a) For $n \geq 1$,

$$n - (1/2) < (n+1) - (1/2) \implies \frac{1}{(n+1) - (1/2)} < \frac{1}{n - (1/2)}$$

so the sequence is decreasing. (What happens when $n = 0$? How can you explain that in light of the above?) Furthermore, For $n \geq 1$, $1/2 \geq n - (1/2)$, so $0 < \frac{1}{n - (1/2)} \leq 2$, so the sequence is bounded below by 0 and above by 2.

(b) Roughly speaking, one would expect that b_n is decreasing because the denominator is of order n^2 which increases faster than the numerator which is of order n . However, we should check the first few terms, which may behave in an exceptional manner: $b_0 = 0$, $b_1 = \frac{2}{5}$, $b_2 = \frac{1}{2}$, $b_3 = \frac{6}{13}$, $b_4 = \frac{4}{9}$, \dots , (use your calculator if you have trouble comparing fractions). Therefore the sequence increases for a time, then decreases, so it is neither uniformly increasing nor decreasing.

(On the other hand, we are usually interested only in what happens for 'large n '. Try to show that the sequence is decreasing for large n .)

As for boundedness, we have (by completing the square)

$$\frac{2n}{n^2 + 4} > k > 0 \implies n^2 - 2\frac{1}{k}n + 4 < 0 \implies \left(n - \frac{1}{k}\right)^2 < \frac{1}{k^2} - 4$$

which is possible only for $k < 1/2$ since the right side of the latter inequality above is nonnegative. Therefore the sequence is bounded above by $1/2$ (you should check that statement), and is clearly bounded below by 0, so it is bounded.

(If the above argument seems strange and backwards to you, you are right: I have shown the ‘analysis’ phase of the argument rather than the proof, which I have asked you to provide. In place of the analysis you could have done some numerical experimentation, but no matter how you arrive at the proposed upper bound of $1/2$, you should check that it really is an upper bound. Another way you could have handled the problem is by minimizing and maximizing the function $f(x) = \frac{2x}{x^2 + 4}$ on the interval $x \geq 0$.)

5. Since I did not specify that we should start at $n = 0$ or $n = 1$, either is fine. Let’s start at $n = 0$ since it’s easier.

(a) Using the fact that $s_n = a_0 + a_1 + \cdots + a_n = s_{n-1} + a_n$, $n \geq 1$ to simplify our calculations,

$$\begin{aligned} s_0 &= a_0 = 1 \\ s_1 &= s_0 + a_1 = 1 - 5 = -4 \\ s_2 &= s_1 + a_2 = -4 + \frac{25}{2} = \frac{17}{2} \\ s_3 &= s_2 + a_3 = \frac{17}{2} - \frac{125}{6} = -\frac{74}{6} \\ s_4 &= s_3 + a_4 = -\frac{74}{6} + \frac{625}{24} = \frac{329}{24}. \end{aligned}$$

(b) Without any simplification at all we can write

$$\begin{aligned} s_0 &= b_0 = 2 \\ s_1 &= s_0 + b_1 = 2 + 2(1-x) \\ s_2 &= s_1 + b_2 = 2 + 2(1-x) + 2(1-x)^2 \\ s_3 &= s_2 + b_3 = 2 + 2(1-x) + 2(1-x)^2 + 2(1-x)^3 \\ s_4 &= s_3 + b_4 = 2 + 2(1-x) + 2(1-x)^2 + 2(1-x)^3 + 2(1-x)^4. \end{aligned}$$

It is possible to simplify the above expressions, and although that is not necessary for this problem, it will be helpful later when we consider Taylor series:

$$\begin{aligned} s_n &= 2 + 2(1-x) + \cdots + 2(1-x)^{n-1} + 2(1-x)^n \\ (1-x)s_n &= 2(1-x) + 2(1-x)^2 + \cdots + 2(1-x)^n + 2(1-x)^{n+1} \\ (1-x)s_n - s_n &= 2(1-x)^{n+1} - 2 \\ s_n &= 2 \frac{(1-x)^{n+1} - 1}{(1-x) - 1} = -\frac{2}{x} ((1-x)^{n+1} - 1). \end{aligned}$$

6. (a) The given series is equal to $\sum_{n=1}^{\infty} \frac{10}{3} \left(\frac{3}{2}\right)^n$ which is geometric with $a = 10/3$, $r = 3/2$; therefore the series diverges because $r > 1$.

(b) The given series is equal to $\sum_{n=1}^{\infty} 10 \left(\frac{-\pi^2}{10}\right)^n$ which is geometric with $a = 10$, $r = -\pi^2/10$. Since $\pi^2 \approx 9.8696 < 10$ we have $-1 < r < 1$ so the series is convergent. The sum is $s = \frac{a}{1-r} = \frac{100}{10 + \pi^2} \approx 5.0323$.

(c) The series can be rewritten $\sum_{n=6}^{\infty} \frac{1}{n}$ which is a tail of the harmonic series so must be divergent.

(d) The series is the sum of two convergent (geometric) series and therefore converges:

$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{2}{8}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{8}\right)^n = \frac{1/4}{1 - (1/4)} + \frac{1/2}{1 - (1/2)} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

7. (a) The series is geometric with $a = 3$, $r = x/4$, therefore it converges for

$$-1 < \frac{x}{4} < 1 \implies -4 < x < 4$$

and diverges otherwise. When it converges the sum of the series is $s = \frac{3}{1 - (x/4)} = \frac{12}{4 - x}$.

- (b) The series is geometric with $a = 1$, $r = 5x/4$, so it converges for $-(4/5) < x < (4/5)$ and diverges otherwise. When it converges the sum is $s = \frac{1}{1 - (5x/4)} = \frac{4}{4 - 5x}$.

- (c) The series is geometric with $a = 2$, $r = 1 - x$, so it converges for

$$-1 < 1 - x < 1 \implies -2 < -x < 0 \implies 0 < x < 2$$

(note the reversal of the inequalities when we multiply through by -1). When it converges the sum is $s = \frac{2}{1 - (1 - x)} = \frac{2}{x}$.

- (d) The series is geometric with $a = 1$, $r = \tan x / \sqrt{3}$, so it converges when

$$-1 < \frac{\tan x}{\sqrt{3}} < 1 \implies -\sqrt{3} < \tan x < \sqrt{3} \implies -\frac{\pi}{3} + k\pi < x < \frac{\pi}{3} + k\pi$$

for $k = 0, \pm 1, \pm 2, \dots$ (It's OK if you didn't get the $k\pi$ part; graph \tan to see where it comes from). When the series converges, its sum is $s = \frac{1}{1 - (\tan x / \sqrt{3})} = \frac{\sqrt{3}}{\sqrt{3} - \tan x}$.

8. A simple rearrangement turns the series into a telescoping series:

$$s_k = \sum_{n=1}^k \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^k \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^k \ln(n+1) - \ln(n).$$

The terms in the middle of a partial sum cancel and we have

$$s_k = \ln(2) - \ln(1) + \ln(3) - \ln(2) + \dots + \ln(k) - \ln(k-1) + \ln(k+1) - \ln(k) = \ln(k+1) - \ln(1) = \ln(k+1).$$

Therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k \rightarrow \infty} \sum_{n=1}^k a_n = \lim_{k \rightarrow \infty} \ln(k+1) = \infty,$$

i.e., the series diverges.