MATH 111 Problem Set 8 Solutions

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1. (a) We have

$$\int x^{-1/3}\,dx = \frac{x^{2/3}}{+}C$$

so the improper integral $\int_1^\infty x^{-1/3}\,dx$ diverges and therefore the given series diverges.

(b) We have

$$\int_1^\infty \frac{x}{x^2+5}\,dx > \int_1^\infty \frac{1}{2x}\,dx$$

which diverges, so by the comparison test for integrals the left hand side of the above also diverges, and therefore the series diverges. (Alternatively, you could have evaluated the integral explicitly using the trig substitution $u = \sqrt{5}\tan\theta$ to show that it diverges.)

(c) Making the substitution $u = x^3$,

$$\int_1^\infty x^2 e^{-x^3}\,dx = \int_1^\infty \frac{1}{3}e^{-u}\,du$$

which converges, so the integral on the left hand side converges, and the series converges.

(d) Making the substitution $u = \ln x$,

$$\int_1^\infty \frac{\ln x}{x}\,dx = \int_0^\infty u\,du$$

which diverges, so the left hand side of the above diverges and the series diverges.

2. (a) The terms of the given series are like $1/n^2$ so we try to compare with something similar. It is easy to see that

$$n^2 < n^2+3n+2 \implies \frac{1}{n^2+3n+2} < \frac{1}{n^2}$$

for n large enough, and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the given series converges by the comparison test.

(b) The terms of the given series are like $1/n^2$ so we try to compare with something similar. Again, it is easy to see that

$$n^4-2n^2 < n^4+1 \implies \frac{n^2-2}{n^4+1} < \frac{1}{n^2}$$

for all n, and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the given series converges by the comparison test.

(c) The terms of the given series are like $5^n/6^n$ so we try to compare with something similar. It is easy to see that

$$5^n 6^n < 5^n + 5^n 6^n \implies \frac{5^n}{1+6^n} < \frac{5^n}{6^n},$$

and the series with the latter terms converges (geometric with $|r| = 5/6 < 1$), so the given series converges by the comparison test.

(d) The terms of the given series are like $1/\sqrt[3]{n}$ so we try to compare with something similar, this time below because we want to show that the given series is divergent.

$$\frac{1}{2}(\sqrt[3]{n}+1) < \sqrt[3]{n} \implies \frac{1}{2\sqrt[3]{n}} < \frac{1}{\sqrt[3]{n}+1}$$

for n large enough, and the series with terms on the left side of the latter inequality above diverges, so the given series diverges as well by the comparison test.

3. (a) The function $f(x) = x^{-1/3}$ decreases for $x > 0$ because its first derivative is negative and $\lim_{x\to\infty} f(x) = 0$ so the absolute values of the terms of the series decrease to zero. By the alternating series test, the series converges.

(b) Consider the function $f(x) = x^2/(2x^3+3)$. Its first derivative is

$$f'(x) = 2x(2x^3+3)^{-1} - x^2(2x^3+3)^{-2}(6x^2) = \frac{2x(2x^3+3) - 6x^4}{(2x^3+3)^2} = \frac{-2x^4+6x}{(2x^3+3)^2}.$$

Since the first derivative is negative for x large enough (the $-x^4$ term in the numerator dominates the numerator for x large, and of course the denominator is positive), f is decreasing for x large enough. Furthermore dividing f through by the highest power of x, $\lim_{x\to\infty} f(x) = 0$. Therefore the absolute values of the terms of the series decrease to 0 for n large enough and the series converges by the alternating series test.

(c) By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x}{(\ln x)^2} = \lim_{x\to\infty} \frac{1}{2(\ln x)/x} = \lim_{x\to\infty} \frac{x}{2\ln x} = \lim_{x\to\infty} \frac{1}{2/x} = \lim_{x\to\infty} \frac{x}{2} = \infty,$$

so the terms of the series do not tend to 0 and the series is divergent by the test for divergence.

(d) By the above calculation, $f(x) = (\ln x)^2/x$ tends to 0 as $x \to \infty$. Furthermore

$$f'(x) = \frac{2\ln x - (\ln x)^2}{x^2}$$

is negative for large x (why?), so the absolute values of the terms of the series decrease to zero for n large enough. By the alternating series test the series converges.

4. (a) I would use the integral test for this series. Integrating by parts,

$$\int x2^{-x}\,dx = -x\frac{2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2}\,dx = -x\frac{2^{-x}}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2} + C$$

(check by differentiating). The limit of the above as $x \to \infty$ so the improper integral converges and therefore the series converges.

(b) We can re-write a term of the series as

$$\frac{n}{1} \cdot \frac{n}{2} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n}.$$

Since each of the multiplicands in the above product is greater than 1, the product as a whole is greater than 1. Therefore the terms of the series do not converge to 0 and the series is divergent by the test for divergence.

5. This looks like a job for the integral test. Make the substitution $u = \ln x$, $du = 1/x\,dx$, $x = e^u$:

$$\int \frac{\ln x}{x^p}\,dx = \int \frac{u}{(e^u)^{p-1}}\,du = \int ue^{-(p-1)u}\,du.$$

Now integrate by parts as in 4a to show that the integral converges for $p > 1$ and diverges for $p \leq 1$.

6. I feel like trying the integral test again. Let $u = \ln x$, $du = 1/x\,dx$, $x = e^u$, $dx = e^u\,du$.

$$\int b^{\ln x}\,dx = \int b^u e^u\,du = \int e^{(\ln b+1)u}\,du.$$

The corresponding improper integral is convergent if and only if $\ln b < -1$, i.e., if $b < 1/e$. However, the series does not make sense for $b < 0$, so the answer is the series is convergent for b satisfying $0 \leq b < 1/e$.