MATH111 200630 Notes

Supplemental Material for Differential Equations

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Identifying First Order DEs

To identify a first order differential equation, first write it in the form

$$\frac{dy}{dx} = F(x, y)$$

where $F(x, y)$ is some function of x and y but does not contain derivatives.
After we solve for dy/dx, we match equation to a pattern that we have learned.

Identifying DEs: Examples

For example, if our differential equation is

$$5x\frac{dy}{dx} + 6xy^2 = 12$$

we can solve for dy/dx and write

$$\frac{dy}{dx} = \frac{12}{5x} - \frac{6}{5}y^2.$$

As another example, if our differential equation is

$$\left(\frac{dy}{dx}\right)^2 = e^{xy}$$

we can solve for dy/dx and write

$$\frac{dy}{dx} = e^{xy/2}.$$

Four Types of First Order DEs

We have learned four types of First Order DE patterns. They are

- Separable: $\dfrac{dy}{dx} = f(x)g(y)$
- Linear: $\dfrac{dy}{dx} = -P(x)y + Q(x)$
- Homogeneous: $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$
- Exact: if none of the other three are appropriate, the equation is probably exact.

Separable Equations

In detail, a separable equation matches the pattern

$$\frac{dy}{dx} = f(x)g(y)$$

where the right hand side is a product of a function of x alone and a function of y alone, i.e., the variables x and y are not too tangled up in the left hand side. For example,

$$\frac{dy}{dx} = x^2y^3$$

is separable, but

$$\frac{dy}{dx} = \sin(xy)$$

is not.

Separable equations are generally easy to identify, but there are some tricky cases.

Tricky Separable Equations I

The equation

$$\frac{dy}{dx} = e^{x+y}$$

is separable, but it is not immediately clear. You should use the laws of exponents to write

$$\frac{dy}{dx} = e^x e^y$$

which is clearly separable.

Tricky Separable Equations II

Another tricky case is the equation

$$\frac{dy}{dx} = 2 + 2y + x + xy$$

which can be identified as separable only after factoring the right hand side:

$$\frac{dy}{dx} = (2 + x)(1 + y)$$

Identifying Linear Equations

Recall that a linear equation matches the pattern

$$\frac{dy}{dx} = -P(x)y + Q(x)$$

For example,

$$\frac{dy}{dx} = xy + e^x$$

is linear, with $P(x) = -x$ and $Q(x) = e^x$.

Tricky Linear Equations

There is no trickery involved in identifying linear equations. The only example which may not be obvious is something like

$$\frac{dy}{dx} = 2y + x$$

where $P(x) = -2$ is considered a function of x but doesn't contain the x variable.

Identifying Homogeneous Equations

A homogeneous DE matches the pattern

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

For example,

$$\frac{dy}{dx} = \cos\left(\frac{y}{x}\right)$$

is homogeneous.

Tricky Homogeneous Equations I

One example of a tricky homogeneous equation is

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}.$$

It's not immediately obvious that it matches the pattern for homogeneous equations, but it does if we write

$$f(v) = v^2 + 2v$$

in which case the above differential equation can be written

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Tricky Homogeneous Equations II

Another tricky homogeneous equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

We can write

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

so the equation is homogeneous with

$$f(v) = v + \frac{1}{v}.$$

A Test for Homogeneous Equations

One way to check whether an equation is homogeneous is to multiply both x and y by some constant k and see whether the right hand side of the equation changes. For example, if the left hand side is

$$F(x,y) = \frac{x^2 + y^2}{xy}$$

then multiplying both x and y by k we have

$$\frac{(kx)^2 + (ky)^2}{(kx)(ky)} = \frac{k^2(x^2 + y^2)}{k^2(xy)} = \frac{x^2 + y^2}{xy}$$

so the ks cancel leaving the left hand side unchanged. We say that F is a homogeneous function: the numerator and denominator are of the same degree in x and y in this case.

Identifying Exact Equations

Exact equations are considerably harder to identify, so we will discuss them later, after we have learned to solve the other three types.

Solving Separable Equations

Once we have identified an equation as separable, i.e.,

$$\frac{dy}{dx} = f(x)g(y),$$

it is easy to solve. The process works as follows: first move all the ys to one side of the equation, and all the xs to the other side, to obtain

$$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Now integrate to obtain

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

Constants of integration will appear when the integrals are evaluated; it's usually best to move them all to the left hand side, so we can write

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$

Solving Separable DEs: Example I

Consider the equation

$$\frac{dy}{dx} = \frac{e^{2x}}{4y^3}.$$

The equation is separable with $f(x) = e^{2x}$ and $g(y) = 1/(4y^3)$.
We write

$$4y^3\, dy = e^{2x}\, dx.$$

Integrating both sides we have

$$\int 4y^3\, dy = \int e^{2x}\, dx$$

$$y^4 = \frac{1}{2}e^{2x} + C.$$

It's nice to solve for y if we can:

$$y = \left(\frac{1}{2}e^{2x} + C\right)^{1/4}.$$

You should check that that family of functions satisfies the differential equation.

Solving Separable DEs: Example II

Consider the tricky separable DE

$$\frac{dy}{dx} + e^{x+y} = 0.$$

We solve for dy/dx and then separate the right hand side:

$$\frac{dy}{dx} = -e^{x+y}$$
$$\frac{dy}{dx} = -e^{x}e^{y}$$

i.e., $f(x) = -e^x$ and $g(y) = e^y$. Solving,

$$e^{-y}\,dy = -e^x\,dx$$
$$\int e^{-y}\,dy = \int -e^x\,dx$$
$$-e^{-y} = -e^x + C.$$

Solving for y so we can check our answer,

$$y = -\ln(e^x + C).$$

Solving Linear Differential Equations

Linear differential equations can be reduced to separable equations, but the process is complicated. It's best just to remember that we need to find an "integrating factor". Given the linear DE

$$\frac{dy}{dx} = -P(x)y + Q(x)$$

we write

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and then multiply by the integrating factor

$$I(x) = e^{\int P(x)\,dx}$$

to obtain

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x).$$

Using the product rule, that equation can be written

$$\frac{d}{dx}(I(x)y) = I(x)Q(x).$$

We then integrate both sides and solve for y.

Solving Linear DEs: Example I

For example, consider the linear differential equation

$$\frac{dy}{dx} = x + 5y.$$

Moving all the ys to the left hand side we have

$$\frac{dy}{dx} - 5y = x.$$

We have $P(x) = -5$ so

$$I(x) = e^{\int P(x)\,dx} = e^{-5x}.$$

Multiplying by the integrating factor,

$$\begin{aligned} e^{-5x}y' - 5e^{-5x}y &= xe^{-5x} \\ \left(e^{-5x}y\right)' &= xe^{-5x} \\ e^{-5x}y &= \int xe^{-5x}\,dx. \end{aligned}$$

We now integrate the right hand side (by parts) and solve for y.

Solving Linear DEs: Example II

Given the equation

$$xy' = \sqrt{x} - y$$

We put it into the standard form

$$y' + \frac{1}{x}y = x^{-1/2}.$$

An integrating factor is

$$I(x) = e^{\int P(x)\,dx} = e^{\ln x} = x.$$

Multiplying by the integrating factor and solving we have

$$\begin{aligned} xy' + y &= x^{1/2} \\ (xy)' &= x^{1/2} \\ xy &= \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C \\ y &= \frac{2}{3}x^{1/2} + Cx^{-1}. \end{aligned}$$

You should check that family solves the equation.

Solving Homogeneous Differential Equations

We can write a homogeneous equation in the standard form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Now make the change of variables $v = y/x$, $y = xv$ and reduce to a separable equation. In detail, by the product rule, the left hand side becomes

$$\frac{d}{dx}y = \frac{d}{dx}(xv) = x\frac{dv}{dx} + v$$

and the right hand side is simply $f(v)$. Therefore the equation becomes

$$x\frac{dv}{dx} + v = f(v)$$

which is separable. Solving as for any separable equation,

$$x\frac{dv}{dx} = f(v) - v$$

$$\frac{1}{f(v) - v}\,dv = \frac{1}{x}\,dx$$

and so on.

Solving Homogeneous DEs: Example I

Consider the homogeneous differential equation

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x}.$$

Making the substitution $y = xv$ we have

$$x\frac{dv}{dx} + v = v^2 + 2v$$
$$1/(v^2 + v)\ dv = 1/x\ dx.$$

Integrating both sides we have

$$\ln|v| - \ln|v+1| = \ln|x| + C.$$

We should solve for v explicitly if we can: we have

$$\ln|v/(v+1)| = \ln|x| + C$$
$$v = (Cx^{-1} - 1)^{-1}.$$

Now we reverse the substitution to obtain

$$y = x(Cx^{-1} - 1)^{-1}.$$

Solving Homogeneous DEs: Example II

Consider the differential equation

$$2xy\frac{dy}{dx} = 3y^2 - x^2 \implies \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}.$$

It is not obvious that the equation is homogeneous, but it is (why?). Substituting $y = vx$ we have

$$x\frac{dv}{dx} + v = \frac{3x^2v^2 - x^2}{2x^2v} = \frac{3v^2 - 1}{2v},$$

separable. Separating variables and substituting $u = v^2 - 1$,

$$x\,\frac{dv}{dx} = \frac{v^2 - 1}{2v} \implies \frac{2v}{v^2 - 1}\,dv = \frac{1}{x}\,dx \implies \ln|u| = \ln|x| + C.$$

Reversing the substitution and solving explicitly for y,

$$v^2 - 1 = Cx \implies \left(\frac{y}{x}\right)^2 = Cx + 1 \implies y = x\sqrt{Cx + 1}.$$

Introduction to Exact Equations

Separable, linear, and homogeneous equations are all special cases of a more general type of differential equation: exact equations. It will take some work before we can define exactly what we mean by exact equations, but the general idea comes from conservation laws in physics. The idea is this: if all ways of going from point A to point B use the same amount of energy, the force field in operation is called conservative, and is described by some exact differential equation.

Partial Differentiation

In order to describe exactly what we mean by an exact equation, we first need to understand the concept of partial differentiation. If a function depends on two variables, for example

$$F(x,y) = x^2 - 2xy + y^2$$

we can make a function of one variable out of it by keeping one of the variables constant and allowing the other to remain variable. For example, if y is fixed and x allowed to vary, we have

$$\frac{\partial}{\partial x} F(x,y) = \frac{\partial}{\partial x} x^2 - 2\frac{\partial}{\partial x} xy + \frac{\partial}{\partial x} y^2 = 2x - 2y,$$

the partial derivative of F with respect to x.

Partial Differentiation II

The special notation

$$\frac{\partial}{\partial x}$$

is used to indicate that the variable y is considered fixed while we differentiate with respect to x. The symbol ∂ is pronounced "di" so the partial differentiation with respect to x operator is read "di by di x". We also have a partial differentiation operator with respect to y. For example,

$$\frac{\partial}{\partial y}(x^2 - 2xy + y^2) = -2x + 2y$$

which is different from the result of partial differentiation with respect to x.

Partial vs. Ordinary Differentiation

We use different notation for partial and for ordinary differentiation because the results of the two operations are different. Recall that

$$\frac{\partial}{\partial x}(x^2 - 2xy + y^2) = 2x - 2y$$

but with ordinary differentiation (or "implicit differentiation" as it is sometimes called) we have

$$\frac{d}{dx}(x^2 - 2xy + y^2) = 2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx}.$$

The results of the two operations are symbolically quite different. Ordinary differentiation gives more information than partial differentiation; the result of ordinary differentiation can be reduced to partial partial differentiation if we take $dy/dx = 0$, i.e., if we hold y fixed. But it's best not to confuse the two ideas.

Partial Differentiation: Example

Consider the function

$$F(x,y) = e^{x^2+3y}.$$

Let's find the partial derivatives of F. First, we consider y to be constant and differentiate with respect to x to obtain

$$\frac{\partial}{\partial x}F(x,y) = \frac{\partial}{\partial x}e^{x^2+3y} = e^{x^2+3y}\frac{\partial}{\partial x}(x^2+3y) = e^{x^2+3y}\cdot 2x$$

by the chain rule. Similarly, the partial with respect to y is

$$\frac{\partial}{\partial y}F(x,y) = \frac{\partial}{\partial y}e^{x^2+3y} = e^{x^2+3y}\frac{\partial}{\partial y}(x^2+3y) = e^{x^2+3y}\cdot 3.$$

Higher Partial Derivatives

Just as we can take higher ordinary derivatives, we can take higher partial derivatives. The possible second partial derivatives in two variables x and y are

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\frac{\partial}{\partial y}.$$

Those operators are usually abbreviated

$$\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial y \partial x}, \frac{\partial^2}{\partial y^2}$$

respectively. There are also third, fourth, etc., partial derivatives, although we will not be using them in our study of exact equations.

Higher Partials: Example

We have the following by application of previous results and the product rule:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}e^{x^2+3y} = \frac{\partial}{\partial x}e^{x^2+3y}\cdot 2x = e^{x^2+3y}(2x)^2 + e^{x^2+3y}\cdot 2$$
$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}e^{x^2+3y} = \frac{\partial}{\partial x}e^{x^2+3y}\cdot 3 = e^{x^2+3y}\cdot 2x\cdot 3$$
$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}e^{x^2+3y} = \frac{\partial}{\partial y}e^{x^2+3y}\cdot 2x = e^{x^2+3y}\cdot 3\cdot 2x$$
$$\frac{\partial}{\partial y}\frac{\partial}{\partial y}e^{x^2+3y} = \frac{\partial}{\partial y}e^{x^2+3y}\cdot 3 = e^{x^2+3y}\cdot 3\cdot 3.$$

Note that most of the results are different, the middle two results are the same.

Equality of Mixed Partials

Higher partial derivatives of the form

$$\frac{\partial^2}{\partial x \partial y} \text{ and } \frac{\partial^2}{\partial y \partial x}$$

are called mixed partial derivatives. Note that in the previous example we had

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}e^{x^2+3y} = e^{x^2+3y} \cdot 2x \cdot 3 = e^{x^2+3y} \cdot 3 \cdot 2x = \frac{\partial}{\partial y}\frac{\partial}{\partial x}e^{x^2+3y}.$$

That is an example of a general principle: the mixed partial derivatives

$$\frac{\partial^2}{\partial x \partial y}F(x,y) = \frac{\partial^2}{\partial y \partial x}F(x,y)$$

are equal for almost all functions $F(x,y)$. There are some highly pathological exceptions which must be carefully constructed, but we will encounter no such functions in our current context. So it is safe to work on the principle that mixed partials are equal.

The same principle applies for higher derivatives and more than two variables, although we will not need to use those generalizations.

The Total Differential, one variable

Recall from MATH 110 that, given a small change Δx in the variable x, a small change Δf in the value of a function f is approximated by

$$\Delta f \approx \frac{df}{dx}\Delta x.$$

We can see that geometrically by approximating f by a tangent line. We can write the above equation as an exact equation if we use "infinitesimals":

$$df = \frac{df}{dx}dx.$$

The symbol df is known as the "total differential of f". It looks as if we are simply cancelling dx on the right hand side, but the actual reasoning involved is considerably more complicated. In any case, the above formula contains an important grain of truth that can be adapted to functions of two variables.

In words, we can say that an (infinitesimally) small change in f is equal to the derivative of f with respect to x times an (infinitesimally) small change in x.

The Total Differential, two variables

We would like a similar way of approximating small changes in a function $F(x,y)$ of two variables. In fact, we have the formula

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

where dF is called the "total differential of F". In words, an (infinitesimally) small change in F is approximated by the partial of F with respect to x times a small change in x, plus the partial of F with respect to y times a small change in y.
The above formula can be derived from geometric considerations: essentially we are approximating the function $F(x,y)$ by its tangent plane at a point.

Total Differentials, Examples

Consider our old friend, the function

$$F(x, y) = e^{x^2+3y}.$$

Its total differential is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 2xe^{x^2+3y}\, dx + 3e^{x^2+3y}\, dy$$

by our previous calculations. As another example, consider the function $G(x, y) = \sin(x^2 - 2xy + y^2)$. Its partials are

$$\frac{\partial G}{\partial x} = \cos(x^2 - 2xy + y^2)(2x - 2y)$$

$$\frac{\partial G}{\partial y} = \cos(x^2 - 2xy + y^2)(-2x - 2y)$$

by the chain rule, so its total differential is

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 2\cos(x^2 - 2xy + y^2)((x - y)dx + (y - x)dy).$$

In general it is quite straightforward to calculate the total differential of a function of two variables.