

# Math 111 Problem Set 1 Solutions DRAFT

Edward Doolittle

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1. Divide through by the fastest-increasing function of  $x$ , i.e.,  $e^x$ , and then apply the limit laws to obtain

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2e^x - 1}{-3e^x + 2} &= \lim_{x \rightarrow \infty} \frac{2e^x/e^x - 1/e^x}{-3e^x/e^x + 2/e^x} \\ &= \lim_{x \rightarrow \infty} \frac{2 - e^{-x}}{-3 + 2e^{-x}} \\ &= \frac{2 - \lim_{x \rightarrow \infty} e^{-x}}{-3 + 2 \lim_{x \rightarrow \infty} e^{-x}}.\end{aligned}$$

We know that  $\lim_{x \rightarrow \infty} e^{-x} = 0$  from the textbook, so the answer is  $2 / -3 = -2/3$ .

2. (a) By the product rule,

$$\begin{aligned}k'(x) &= \left( \frac{d}{dx}(x^3 + 1) \right) e^x + (x^3 + 1) \frac{d}{dx} e^x \\ &= 3x^2 e^x + (x^3 + 1) e^x \\ &= (x^3 + 3x^2 + 1) e^x.\end{aligned}$$

- (b) By the chain rule,

$$\begin{aligned}m'(x) &= \sin'(e^{x^2}) \frac{d}{dx} e^{x^2} \\ &= \cos(e^{x^2}) e^{x^2} \frac{d}{dx} x^2 \\ &= \cos(e^{x^2}) e^{x^2} 2x.\end{aligned}$$

3. (a) Let  $u = 2x$ . Then  $du = 2 dx$  and

$$\begin{aligned}\int e^{2x} dx &= \int e^u \frac{du}{2} \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{2x} + C.\end{aligned}$$

Check by differentiating.

- (b) Let  $u = 2x^2$ . Then  $du = 4x dx$  and

$$\begin{aligned}\int x e^{2x^2} dx &= \int e^u \frac{du}{4} \\ &= \frac{1}{4} e^u + C \\ &= \frac{1}{4} e^{2x^2} + C.\end{aligned}$$

As usual, check by differentiating.

4. First, find the  $y$ -value of the point on the curve. When  $x = \pi$ ,  $y = e^x \sin(x) = e^\pi \sin(\pi) = e^\pi \cdot 0 = 0$ . So the point on the curve is  $(\pi, 0)$ . Next find the slope of the tangent line through that point. By the product rule, the derivative of the function is  $y' = e^x \sin(x) + e^x \cos(x)$ . So the derivative for the given value of  $x$  is  $y'(x) = y'(\pi) = e^\pi \sin(\pi) + e^\pi \cos(\pi) = e^\pi \cdot 0 + e^\pi \cdot -1 = -e^\pi$ . That is the slope  $m$  of the tangent line. Since we know a point through which the line passes and the slope of the line, we can write down an equation in point-slope form:  $y - 0 = -e^\pi(x - \pi)$ . No need to simplify any further.

5. We tackle this problem in stages.

(a) Differentiating,  $f'(x) = e^{2x-x^2}(2-2x)$ . Setting the first derivative equal to 0,  $e^{2x-x^2}(2-2x) = 0$ . Dividing through by  $e^{2x-x^2}$  (which is possible because exponentials are never equal to 0 and in fact are always positive) gives the linear equation  $2-2x=0$  which implies  $x=1$ . So a candidate for the absolute maximum value of the function occurs when  $x=1$ . To check that that is the absolute maximum value, use the first derivative test. When  $x < 1$ ,  $2-2x > 0$  so  $f'(x) = e^{2x-x^2}(2-2x) > 0$  so the function is increasing. When  $x > 1$ ,  $2-2x < 0$  so  $f'(x) = e^{2x-x^2}(2-2x) < 0$  so the function is decreasing. The function increases to  $x=1$  and decreases thereafter, so the absolute maximum value of the function must occur when  $x=1$ . The value is  $f(1) = e^{2-1} = e$ .

(b) Differentiating again by the product rule and chain rule,  $f''(x) = e^{2x-x^2}(2-2x)^2 + e^{2x-x^2}(-2) = e^{2x-x^2}(4-8x+4x^2-2) = e^{2x-x^2}(4x^2-8x+2)$ . Similar to the previous problem,  $f''(x) = 0$  implies

$$2x^2 - 4x + 1 = 0. \tag{1}$$

By the quadratic formula,

$$x = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{2}}{2}.$$

Calculator says  $x_1 = 0.3$  and  $x_2 = 1.7$  approximately. Furthermore  $2x_1^2 - 4x_1 + 1 = 0$  and  $2x_2^2 - 4x_2 + 1 = 0$  because  $x_1$  and  $x_2$  satisfy the quadratic equation (1); dividing by 2 and rearranging gives  $2x_1 - x_1^2 = 1/2$  and  $2x_2 - x_2^2 = 1/2$ . Therefore  $f(x_1) = e^{2x_1-x_1^2} = e^{1/2}$  and  $f(x_2) = e^{2x_2-x_2^2} = e^{1/2}$ ;  $f(x_1) = f(x_2) = 1.6$  approximately, so the inflection points are  $(x_1, f(x_1)) = (0.3, 1.6)$  and  $(x_2, f(x_2)) = (1.7, 1.6)$  to one decimal point of accuracy.

(c) The only information still required is the intervals where the function is concave up/down. The sign of  $f''(x)$  is the same as the sign of  $2x^2 - 4x + 1$ . Since the coefficient of  $x^2$  is positive,  $2x^2 - 4x + 1$  is negative between the two roots and positive elsewhere. (See figure 1, which isn't a required part of the answer, but is just there to give you a sense of how  $2x^2 - 4x + 1$  changes sign; you can also use algebra to explore the sign changes.) Therefore  $f''(x) > 0$  on the intervals  $(-\infty, 0.3)$  and  $(1.7, \infty)$ , so the function  $f(x)$  is concave up on those intervals; and  $f''(x) < 0$  on the interval  $(0.3, 1.7)$ , so the function  $f(x)$  is concave down on those intervals.

Assembling all the information we have gathered gives the graph in figure 2.

6. (a) The graph of the function has equation  $y = (2x-1)/(-3x+2)$ . Therefore the graph of the inverse has equation  $x = (2y-1)/(-3y+2)$ . To put that equation into a form which resembles a function, we must solve for  $y$ :

$$\begin{aligned} (-3y+2)x &= 2y-1 \\ 2x+1 &= 2y+3yx \\ \frac{2x+1}{2+3x} &= y, \end{aligned}$$

so the inverse function is

$$f^{-1}(x) = \frac{2x+1}{2+3x}.$$

To be totally complete and correct you should give the domain of  $f^{-1}$ . You should check your work by verifying that the cancellation equations  $f^{-1}(f(x)) = x$  and  $f(f^{-1}(x)) = x$  hold on the appropriate domains.

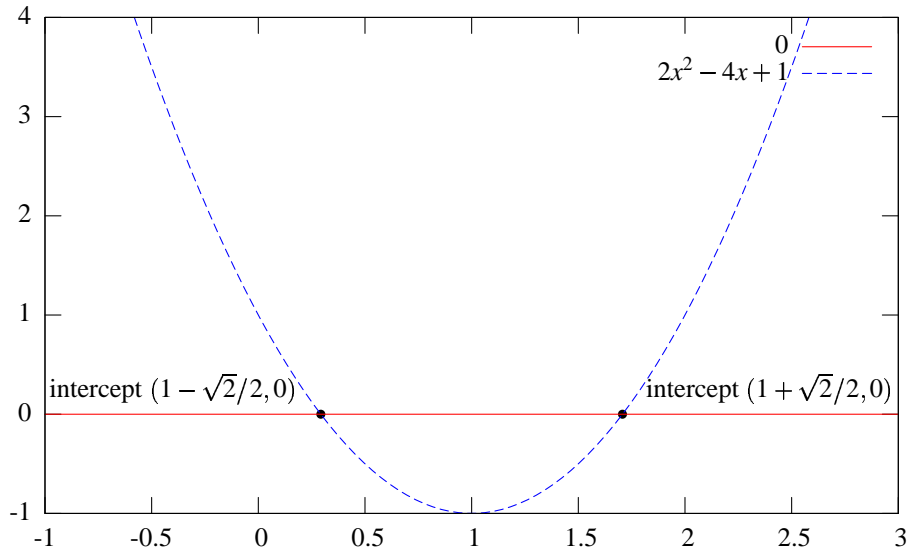


Figure 1: Graph of  $2x^2 - 4x + 1$

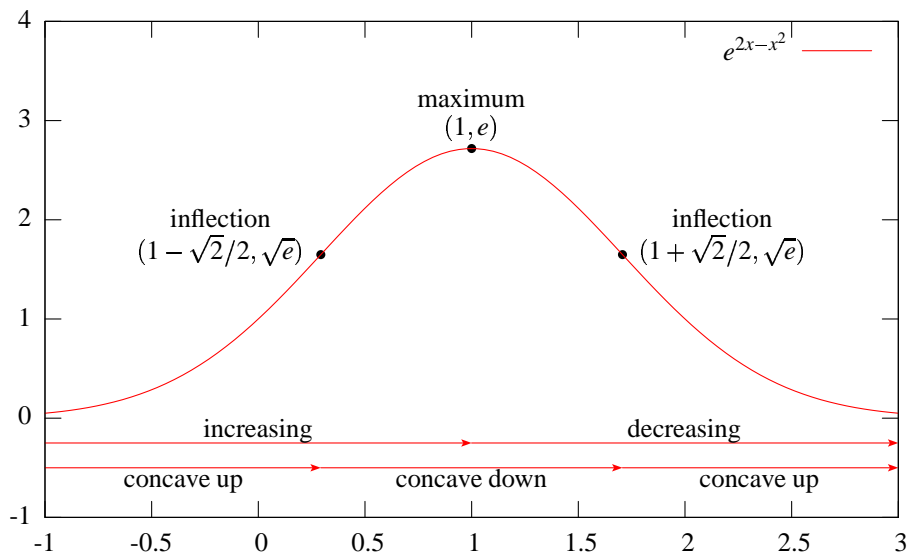


Figure 2: Graph of  $e^{2x-x^2}$

- (b) The graph of the function has equation  $y = (x + 2)^5$ , so the graph of the inverse function has equation  $x = (y + 2)^5$ . To put that equation into function form we should solve for  $y$ :

$$\begin{aligned}x^{1/5} &= y + 2 \\ y &= x^{1/5} - 2\end{aligned}$$

so the inverse function is

$$f^{-1}(x) = x^{1/5} - 2.$$

7. (a)  $h'(x) = 3x^2 + 2 > 0$  for all  $x$  (because  $x^2 \geq 0$  for all  $x$  and  $2 > 0$ ), so the function  $h$  is increasing for all  $x$ , so it must be 1-1.  
 (b) Guessing,  $h(1) = 1^3 + 2(1) + 3 = 1 + 2 + 3 = 6$ , so  $h^{-1}(6) = 1$ . By the formula for derivatives of inverse functions,

$$(h^{-1})'(6) = \frac{1}{h'(h^{-1}(6))} = \frac{1}{h'(1)} = \frac{1}{5}.$$

8. Let's find the first few derivatives and try to find a pattern:

$$\begin{aligned}p^{(1)}(x) &= e^{-x} + (x + 50)e^{-x}(-1) = (-x - 49)e^{-x} \\ p^{(2)}(x) &= -e^{-x} + (-x - 49)e^{-x}(-1) = (x + 48)e^{-x} \\ p^{(3)}(x) &= e^{-x} + (x + 48)e^{-x}(-1) = (-x - 47)e^{-x} \\ p^{(4)}(x) &= -e^{-x} + (-x - 47)e^{-x}(-1) = (x + 46)e^{-x}\end{aligned}$$

and so on. A good guess at the formula would be  $p^{(k)} = (-1)^k(x + 50 - k)e^{-x}$ . (You can prove that formula by induction, but that's not necessary in this course.) So when  $k = 100$ ,  $p^{(100)} = (-1)^{100}(x + 50 - 100)e^{-x} = (x - 50)e^{-x}$ .

9. Let's warm up by proving two simpler results first. We assume that  $e^x > 0$  for all  $x$ .

- (a) Theorem:  $e^x > 1$  for  $x > 0$ . Proof: Form the function  $f(x) = e^x - 1$ . Then  $f'(x) = e^x > 0$  for all  $x$  so  $f$  is increasing for  $x > 0$ . Furthermore  $f(0) = e^0 - 1 = 1 - 1 = 0$  so  $f$  increases above 0 for  $x > 0$ , i.e.,  $f(x) = e^x - 1 > 0$  for  $x > 0$ , i.e.,  $e^x > 1$  for  $x > 0$ .  
 (b) Theorem:  $e^x > x + 1$  for  $x > 0$ . Proof: We prove this in a way similar to that of the previous result. Let  $g(x) = e^x - (x + 1)$ . Then  $g'(x) = e^x - 1$ , and we know that  $e^x - 1 > 0$  for  $x > 0$  from the previous result, so  $g'(x) > 0$  for  $x > 0$  which implies that  $g(x)$  is increasing for  $x > 0$ . Also,  $g(0) = e^0 - 0 - 1 = 1 - 1 = 0$ , so  $g(x)$  is increasing above 0 for  $x > 0$ , i.e.,  $g(x) = e^x - (x + 1) > 0$  for  $x > 0$ , i.e.,  $e^x > x + 1$  for  $x > 0$ .  
 (c) Now we are ready for the given problem. Theorem:  $e^x > x^2/2 + x + 1$  for all  $x > 0$ . Proof: Let  $h(x) = e^x - (x^2/2 + x + 1)$ . Then  $h'(x) = e^x - x - 1$ . By the previous result,  $h'(x) > 0$  for  $x > 0$  so  $h(x)$  is increasing for  $x > 0$ . Furthermore,  $h(0) = e^0 - 0^2/2 - 0 - 1 = 1 - 1 = 0$  so  $h(x)$  is increasing above 0 for  $x > 0$ , i.e.,  $h(x) = e^x - (x^2/2 + x + 1) > 0$  for  $x > 0$ , i.e.,  $e^x > x^2/2 + x + 1$  for  $x > 0$ .  
 (d) Now we use the previous result to estimate the integral. We have

$$1 + x + \frac{x^2}{2} \leq e^x \leq e^1$$

for  $0 \leq x \leq 1$ . Substituting  $x^2$  for  $x$ ,

$$1 + x^2 + \frac{x^4}{2} \leq e^{x^2} \leq e^1$$

for  $0 \leq x^2 \leq 1$ , i.e., for  $0 \leq x \leq 1$ . (In fact, the result holds for  $-1 \leq x \leq 0$  too; why?)

Integrating throughout the previous inequality we have

$$\int_0^1 1 + x^2 + \frac{x^4}{2} dx \leq \int_0^1 e^{x^2} dx \leq \int_0^1 e^1 dx,$$

i.e.,

$$x + \frac{x^3}{3} + \frac{x^5}{10} \Big|_0^1 \leq \int_0^1 e^{x^2} dx \leq e^{1 \cdot x} \Big|_0^1.$$

Evaluating the integrals, we have

$$1 + \frac{1}{3} + \frac{1}{10} \leq \int_0^1 e^{x^2} dx \leq e$$

which shows that the given integral lies between 1.43 and 2.71, with an error of 1.28 or so.