Math 111 Problem Set 2 Solutions

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September 27, 2006

1. (a) $2\log_{10}3 - \log_{10}90 = \log_{10}3^2 - \log_{10}90 = \log_{10}9 - \log_{10}90 = \log_{10}(9/90) = \log_{10}10^{-1} = -1$.

   (b) $\ln(e^{2^3}) = 2^3\ln(e) = 8\cdot 1 = 8$.

2. (a) Exponentiating both sides,

$$\begin{aligned} e^{\ln(5x-2)} &= e^{-7} \\ 5x-2 &= e^{-7} \\ 5x &= e^{-7}+2 \\ x &= \frac{1}{5}(e^{-7}+2). \end{aligned}$$

   My calculator tells me that $x = 0.4002$ to 4 decimal places.

   (b) Taking the log base 2 of both sides,

$$\begin{aligned} \log_2(2^{3x+4}) &= \log_2(10) \\ (3x+4)\log_2(2) &= \log_2(10) \\ 3x+4 &= \log_2(10) \\ 3x &= \log_2(10)-4 \\ x &= \frac{1}{3}(\log_2(10)-4) \end{aligned}$$

   Since there is no $\log_2$ button on my calculator I have to use the chain rule for logarithms to evaluate $\log_2(10) = \ln(10)/\ln(2)$; with that, my calculator tells me $x = -0.2260$.

3. (a) Write $f(x) = (x\ln x)^{1/2}$. Then by the chain rule and product rule,

$$\begin{aligned} f'(x) &= \frac{1}{2}(x\ln x)^{-1/2}\frac{d}{dx}(x\ln x) \\ &= \frac{1}{2}(x\ln x)^{-1/2}\left(\left(\frac{d}{dx}x\right)\ln x + x\frac{d}{dx}\ln x\right) \\ &= \frac{1}{2}(x\ln x)^{-1/2}\left(\ln x + x\frac{1}{x}\right) \\ &= \frac{1}{2}(x\ln x)^{-1/2}(\ln x+1). \end{aligned}$$

   (b) Rewrite 2^u in terms of the (natural) exponential function: $2^u = (e^{\ln 2})^u = e^{(\ln 2)u}$. Then by the chain rule,

$$\begin{aligned} g'(\theta) &= \frac{d}{d\theta}e^{(\ln 2)\sin\theta} \\ &= e^{(\ln 2)\sin\theta}\frac{d}{d\theta}((\ln 2)\sin\theta) \\ &= 2^{\sin\theta}(\ln 2)\cos\theta. \end{aligned}$$

4. (a) Take the logarithm of both sides:

$$\begin{aligned}\ln y &= \ln\left((2x^3+1)^4(x-1)^{-1/2}\right)\\ &= 4\ln(2x^3+1) - \frac{1}{2}\ln(x-1)\end{aligned}$$

Then by implicit differentiation on the left side and the chain rule on the right side,

$$\begin{aligned}\frac{y'}{y} &= 4\frac{1}{2x^3+1}\frac{d}{dx}(2x^3+1) - \frac{1}{2}\frac{1}{x-1}\frac{d}{dx}(x-1)\\ &= \frac{24x^2}{2x^3+1} - \frac{1}{2x-2}\end{aligned}$$

Multiplying both sides by y and substituting what we know about y,

$$\begin{aligned}y' &= y\left(\frac{24x^2}{2x^3+1} - \frac{1}{2x-2}\right)\\ &= (2x^3+1)^4(x-1)^{-1/2}\left(\frac{24x^2}{2x^3+1} - \frac{1}{2x-2}\right).\end{aligned}$$

Further simplification is possible, but not necessary unless the question asks for it.

(b) Take the logarithm of both sides:

$$\begin{aligned}\ln h &= \ln\left(\frac{\sin^2 x\cos^4 x}{(x^2+1)^3}\right)\\ &= 2\ln\sin x + 4\ln\cos x - 3\ln(x^2+1).\end{aligned}$$

Differentiate the above equation (by implicit differentiation on the left side and the chain rule on the right side):

$$\begin{aligned}\frac{h'}{h} &= 2\frac{1}{\sin x}\cos x + 4\frac{1}{\cos x}(-\sin x) - 3\frac{1}{x^2+1}(2x)\\ &= \frac{2\cos x}{\sin x} - \frac{4\sin x}{\cos x} - \frac{6x}{x^2+1}.\end{aligned}$$

Multiplying by h,

$$h'(x) = \frac{\sin^2 x\cos^4 x}{(x^2+1)^3}\left(\frac{2\cos x}{\sin x} - \frac{4\sin x}{\cos x} - \frac{6x}{x^2+1}\right).$$

5. Differentiating both sides of the equation, then applying the chain rule,

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}e^{x^2+y^2}\\ 2y\frac{d}{dx}y &= e^{x^2+y^2}\frac{d}{dx}(x^2+y^2)\\ 2yy' &= e^{x^2+y^2}(2x+2yy').\end{aligned}$$

Solving for y',

$$\begin{aligned}2yy' - 2ye^{x^2+y^2}y' &= 2xe^{x^2+y^2}\\ 2y(1-e^{x^2+y^2})y' &= 2xe^{x^2+y^2}\\ y' &= \frac{2xe^{x^2+y^2}}{2y(1-e^{x^2+y^2})}\\ &= \frac{xe^{x^2+y^2}}{y(1-e^{x^2+y^2})}.\end{aligned}$$

6. Let I be the required result. First expand the integrand using the formula $(a+b)^2 = a^2+2ab+b^2$:

$$I = \int_4^{16} \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)^2 \, dx = \int_4^{16} \left(x + 6 + \frac{9}{x}\right) \, dx.$$

   Integrating each term,

$$\begin{aligned} I &= \int_4^{16} x\,dx + \int_4^{16} 6\,dx + \int_4^{16} \frac{9}{x}\,dx \\ &= \frac{x^2}{2}\Big|_4^{16} + 6x\Big|_4^{16} + 9\ln x\Big|_4^{16} \\ &= 128 - 8 + 96 - 24 + 9\ln(16) - 9\ln(4) \\ &= 192 + 18\ln(2). \end{aligned}$$

   To four decimal places, $I = 204.4766$.

7. (a) To find the inverse, solve for the independent variable t:

$$\begin{aligned} Q &= Q_0(1 - e^{-t/a}) \\ \frac{Q}{Q_0} &= 1 - e^{-t/a} \\ 1 - \frac{Q}{Q_0} &= e^{-t/a} \\ \ln(1 - Q/Q_0) &= \ln(e^{-t/a}) = -t/a \\ -a\ln(1 - Q/Q_0) &= t, \end{aligned}$$

   so the inverse function is

$$Q^{-1}(q) = -a\ln(1 - q/Q_0).$$

   $Q^{-1}(q)$ should be interpreted as the amount of time it takes for the camera to reach charge level q.

   (b) By the previous result, we need to find $Q^{-1}(0.75Q_0)$:

$$\begin{aligned} Q^{-1}(0.75Q_0) &= -3\ln(1 - (0.75Q_0)/Q_0) \\ &= -3\ln(1 - 0.75) \\ &= -3\ln(1/4) \\ &= 6\ln(2). \end{aligned}$$

   Calculator says $t = 4.2$ seconds approximately.

8. The slope of the line $x + 2y = 5$ is $-1/2$. The slope of any perpendicular line is the negative reciprocal, i.e., 2. We need to find a line tangent to $y = e^x$ with slope 2. The line tangent to the curve at the point (x, e^x) has slope $\frac{d}{dx}e^x = e^x$, so we need to solve the equation $e^x = 2$: $x = \ln(2)$. A the point with the given x-value has coordinates $(\ln(2), e^{\ln(2)}) = (\ln(2), 2)$. The point-slope form of the tangent line is $y - 2 = 2(x - \ln(2))$.

9. Using the hint and the rules for exponents,

$$\begin{aligned} \lim_{x\to 0^+} x^{\ln x} &= \lim_{x\to 0^+} (e^{\ln x})^{\ln x} \\ &= \lim_{x\to 0^+} e^{(\ln x)^2} \\ &= e^{\lim_{x\to 0^+} (\ln x)^2}. \end{aligned}$$

   As $x \to 0^+$, $\ln x \to -\infty$, so $(\ln x)^2 \to \infty$. Therefore $\lim_{x\to 0^+} x^{\ln x} = e^\infty = \infty$.

10. Here are two different solutions:

(a) By formula 8 of section 7.4 and the continuity of the power function $f(x) = x^a$,

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{a}{n}\right)^{n/a}\right)^a = \lim_{h\to 0^+}\left((1+h)^{1/h}\right)^a = \left(\lim_{h\to 0^+}(1+h)^{1/h}\right)^a = e^a$$

where $h = a/n$.

(b) By the continuity of the natural logarithm function,

$$\ln\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\ln\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} n\ln(1+a/n) = \lim_{n\to\infty}\frac{\ln(1+a/n)}{1/n} = a\lim_{n\to\infty}\frac{\ln(1+a/n)}{a/n}.$$

Letting $h = a/n$ the limit becomes

$$a\lim_{n\to\infty}\frac{\ln(1+a/n)}{a/n} = a\lim_{h\to 0^+}\frac{\ln(1+h)}{h} = a\cdot 1 = a$$

by the result of problem 87 of section 7.4. Therefore

$$\ln\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = a$$

and exponentiating both sides,

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$$

as required.