MATH 111 Problem Set 3 Solutions DRAFT

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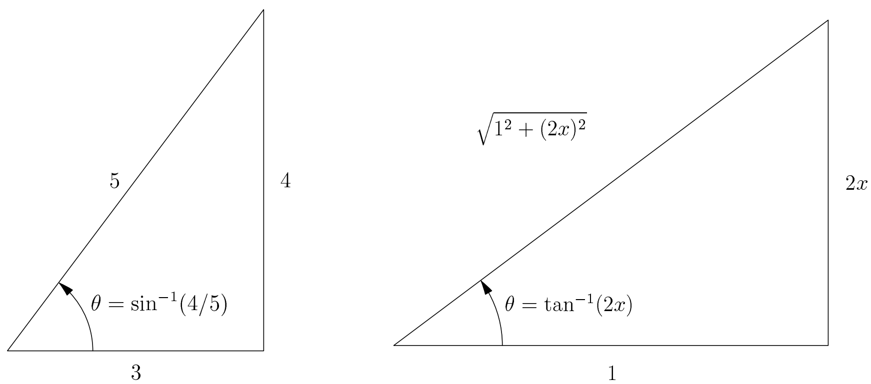
1. (a) In a right triangle with angle $\theta = \sin^{-1}(4/5)$, the opposite side O and hypoteneuse H are 4 and 5 units respectively. Then the adjacent side $A = \sqrt{5^2 - 4^2} = \sqrt{9} = 3$ units. Then $\sec(\sin^{-1}(4/5)) = \sec(\theta) = H/A = 5/3$. See Figure 1(a). (Question: is $A = -3$ possible? Why or why not?)



(a) 3, 4, 5 right triangle (b) 1, $2x$, $\sqrt{1+4x^2}$ right triangle

Figure 1: Two right triangles

   (b) In a right triangle with angle $\theta = \tan^{-1}(2x)$, the opposite side O and adjacent side A are $2x$ and 1 units respectively. Then the hypoteneuse is $H = \sqrt{(2x)^2 + 1^2} = \sqrt{1+4x^2}$ units, and $\sin(\tan^{-1}(2x)) = \sin(\theta) = O/H = 2x/\sqrt{1+4x^2}$. See Figure 1(b). (Question: what happens if x is negative? Can the adjacent side be -1 instead of 1?)

2. (a) The limit is of the form $0/0$. Applying L'Hôpital's rule,

$$\lim_{t\to 0} \frac{e^t - 2^t}{\tan t} = \lim_{t\to 0} \frac{e^t - (\ln 2)2^t}{\sec^2 t} = \frac{1 - \ln 2}{1}$$

   which is 0.3069 to four decimal places.

   (b) Factoring the denominator,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3 \tan^2 x + x^5} = \lim_{x\to 0} \frac{\sin x - x}{x^3} \cdot \frac{1}{\tan^2 x + x^2}.$$

   The first factor is of the form $0/0$. Applying L'Hôpital's rule until it is no longer of that form,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

   The second factor is of the form $1/0$, but we must be careful before we conclude that it is ∞. As $x \to 0$, $\tan^2 x + x^2 \to 0+$ so $1/(\tan^2 x + x^2)$ tends to $+\infty$.

   You should also try this problem with x^3 instead of x^5 in the denominator.

(c) This limit is of the form $\infty \cdot 0$. Dividing through by $1-5x$ and applying L'Hôpital's rule,

$$\lim_{x\to\infty}(1-5x)\sin(1/x) = \lim_{x\to\infty}\frac{\sin(1/x)}{(1-5x)^{-1}} = \lim_{x\to\infty}\frac{\cos(1/x)\cdot(-1/x^2)}{5(1-5x)^{-2}}.$$

Now we try to evaluate the limit using limit laws and algebra:

$$\lim_{x\to\infty}\frac{\cos(1/x)\cdot(-1/x^2)}{5(1-5x)^{-2}}. = \lim_{x\to\infty}\cos(1/x)\cdot\lim_{x\to\infty}\frac{-1/x^2}{5(1-5x)^{-2}}$$
$$= 1\cdot\lim_{x\to\infty}\frac{-(1-5x)^2}{5x^2} = \lim_{x\to\infty}\frac{-((1/x)-5)^2}{5} = -5$$

(d) This limit is of the form 1^∞. Let L be the desired limit. Taking logarithms,

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{2}{x}+\frac{3}{x^2}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}+\frac{3}{x^2}\right).$$

This new limit is of the form $\infty\cdot 0$. Moving x to the denominator (where it becomes x^{-1}) and applying L'Hôpital's rule,

$$\ln L = \lim_{x\to\infty}\frac{\ln(1+2x^{-1}+3x^{-2})}{x^{-1}} = \lim_{x\to\infty}\frac{(1+2x^{-1}+3x^{-2})^{-1}\cdot(-2x^{-2}-6x^{-3})}{-x^{-2}}.$$

Multiplying the second factor and the denominator through by $-x^2$,

$$\ln L = \lim_{x\to\infty}(1+2x^{-1}+3x^{-2})^{-1}\cdot(2+6x^{-1}) = (1+0+0)^{-1}\cdot(2+0) = 2.$$

Therefore $L = e^2 = 7.3891$ to four decimal places.

3. (a) By the chain rule,

$$f'(x) = \arcsin'(e^x)\cdot\frac{d}{dx}e^x = \frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}.$$

(b) Again by the chain rule,

$$h'(z) = (\cot^{-1})'(e^z)\frac{d}{dz}e^z + (\cot^{-1})(e^{-z})\frac{d}{dz}e^{-z} = -\frac{1}{1+(e^z)^2}e^z - \frac{1}{1+(e^{-z})^2}(-e^{-z}).$$

(You can stop there if you like, but a little more algebra gives an interesting result:

$$h'(z) = -\frac{e^z}{1+e^{2z}} + \frac{e^{-z}}{1+e^{-2z}}\cdot\frac{e^{2z}}{e^{2z}} = -\frac{e^z}{1+e^{2z}} + \frac{e^z}{e^{2z}+1} = 0.$$

In fact, $h(z) = \pi/2$, a constant. Question: is the function $k(x) = \cot^{-1}(x) + \cot^{-1}(1/x)$ constant? Careful!)

(c) By the chain rule,

$$g'(t) = (\cos^{-1})'(3-2t)\cdot(-2) = \frac{2}{\sqrt{1-(3-2t)^2}} = 2(1-(3-2t)^2)^{-1/2}.$$

Differentiating again, by the chain rule,

$$g''(t) = -(1-(3-2t)^2)^{-3/2}\cdot\frac{d}{dt}(1-(3-2t)^2) = -(1-(3-2t)^2)^{-3/2}\cdot(-2(3-2t)(-2)).$$

Further simplification is only useful if you are taking a third derivative or doing something else with the function.

(d) By implicit differentiation,

$$\frac{d}{dx}\tan^{-1}(xy) = \frac{d}{dx}(1+x^2y^2).$$

By the chain rule,

$$\frac{1}{1+(xy)^2}\frac{d}{dx}(xy) = \frac{d}{dx}(x^2y^2).$$

By the product rule,

$$\frac{1}{1+x^2y^2}(y+xy') = 2xy^2+2x^2yy'.$$

Solving for y',

$$\begin{aligned}\frac{x}{1+x^2y^2}y' - 2x^2yy' &= 2xy^2 - \frac{y}{1+x^2y^2}\\ y' &= \frac{2xy^2 - y/(1+x^2y^2)}{x/(1+x^2y^2) - 2x^2y}.\end{aligned}$$

4. (a) Dividing through by 9,

$$\int \frac{3}{t^2+9}\,dt = \frac{1}{3}\int \frac{1}{(t/3)^2+1}\,dt.$$

Let $u = t/3$; then $du = dt/3$, $dt = 3du$, and

$$\int \frac{3}{t^2+9}\,dt = \frac{1}{3}\int \frac{3du}{u^2+1} = \arctan(u)+C = \arctan(t/3)+C.$$

(After reading chapter 8.3, it is more natural to make the substitution $t = 3\tan\theta$.)

(b) This is a case where we have to be careful of the sign of the square root. Recall that $\sqrt{u^2} = |u|$, the absolute value of u. Therefore the integrand is

$$\frac{\sin x}{\sqrt{1-\cos^2 x}} = \frac{\sin x}{\sqrt{\sin^2 x}} = \frac{\sin x}{|\sin x|} = \begin{cases} -1, & -\frac{\pi}{4} \le x < 0 \\ +1, & 0 < x \le \frac{\pi}{4} \end{cases}.$$

On the interval in question the above function corresponds to the Heaviside step function $H(x) = x/|x|$ (see Figure 2). Informally, since H is an odd function ($H(-x) = -H(x)$), its integral over any interval symmetric about the origin must be 0. A more formal way of evaluating the integral is to break up the domain of integration into pieces:

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \frac{\sin x}{\sqrt{1-\cos^2 x}}\,dx &= \int_{-\pi/4}^{0} \frac{\sin x}{\sqrt{1-\cos^2 x}}\,dx + \int_{0}^{\pi/4} \frac{\sin x}{\sqrt{1-\cos^2 x}}\,dx\\ &= \int_{-\pi/4}^{0} -1\,dx + \int_{0}^{\pi/4} 1\,dx\\ &= -\pi/4+\pi/4 = 0.\end{aligned}$$

(Actually, $\sin x/|\sin x|$ is a ‘square wave’. Graph it on the larger interval $-3\pi \le x \le 3\pi$ to see why it is called a square wave.)

(c) The integral can be rewritten

$$\int \frac{x+4}{x^2+16}\,dx = \int \frac{x}{x^2+16}\,dx + \int \frac{4}{x^2+16}\,dx. \tag{1}$$

The first term on the left side of (1) can be evaluated by making the substitution $u = x^2+16$, $du = 2x\,dx$, $du/2 = x\,dx$. Then

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln(x^2+16)+C.$$

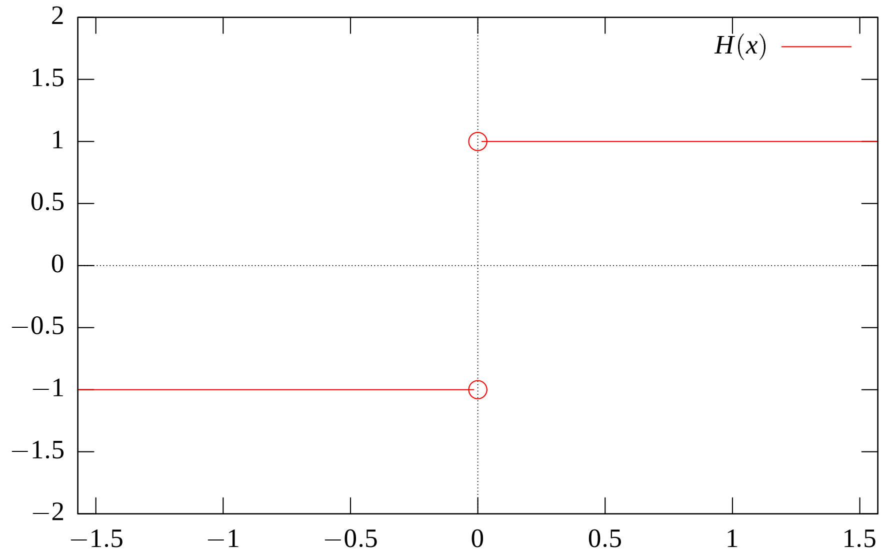


Figure 2: Graph of the Heaviside step function $H(x)$

The second term on the left side of (1) can be evaluated by dividing through by 16 and making the substitution $v = x/4$, $dv = dx/4$, $dx = 4dv$:

$$\int \frac{4}{x^2+16}\,dx = \frac{1}{4}\int \frac{1}{(x/4)^2+1}\,dx = \int \frac{dv}{v^2+1} = \arctan(v)+C = \arctan(x/4)+C.$$

Altogether,

$$\int \frac{x+4}{x^2+16}\,dx = \ln\sqrt{x^2+16}+\arctan(x/4)+C.$$

(d) Let $u = x^3$. Then $du = 3x^2\,dx$ so $x^2\,dx = du/3$ and

$$\int \frac{2x^2}{\sqrt{1-x^6}}\,dx = \frac{2}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{2}{3}\arcsin(u)+C = \frac{2}{3}\arcsin(x^3)+C.$$

5. (a) Multiplying the given limit through by the conjugate radical and then dividing through by x,

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x) = \lim_{x\to\infty}(\sqrt{x^2+2x}-x)\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty}\frac{(x^2+2x)-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty}\frac{2}{\sqrt{1+2/x}+1} = 1.$$

(b) Taking out a factor of x, putting it in the denominator, and then applying L'Hôpital's rule,

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x) = \lim_{x\to\infty}\frac{\sqrt{1+2/x}-1}{1/x} = \lim_{x\to\infty}\frac{(1/2)(1+2/x)^{-1/2}(-2/x^2)}{-1/x^2} = \lim_{x\to\infty}(1+2/x)^{-1/2} = 1.$$

6. This limit is of the form $0/0$ because $\int_0^0 f(t)\,dt = 0$. Applying L'Hôpital's rule and the Fundamental Theorem of Calculus,

$$\lim_{x\to 0}\frac{\int_0^x \tan(t^2)\,dt}{x^3} = \lim_{x\to 0}\frac{\tan(x^2)}{3x^2} = \lim_{x\to 0}\frac{\sec^2(x^2)2x}{6x} = \lim_{x\to 0}\frac{\sec^2(x^2)}{3} = \frac{1}{3}$$

7. The limit is of the form 1^∞. Taking logarithms of both sides,

$$\ln\left(\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x\right) = \ln e,$$

i.e.,

$$\lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = 1.$$

Dividing through by x and applying logarithm laws and L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln((x+a)/(x-a))}{x^{-1}} = \lim_{x\to\infty} \frac{\ln(x+a)-\ln(x-a)}{x^{-1}} = \lim_{x\to\infty} \frac{\frac{1}{x+a}-\frac{1}{x-a}}{-x^{-2}} = \lim_{x\to\infty} \frac{2ax^2}{x^2-a^2} = 2a$$

so the equation becomes $2a = 1$, i.e., $a = 1/2$.