MATH111-002 200630 Problem Set 5 Solutions DRAFT

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1. (a) Make the substitution $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta\,d\theta$:

$$\int \frac{\sqrt{x^2-16}}{x^4}\,dx = \int \frac{\sqrt{16\sec^2\theta-16}}{4^4\sec^4\theta}4\sec\theta\tan\theta\,d\theta = \int \frac{4\tan\theta}{4^3\sec^3\theta}\tan\theta\,d\theta = \frac{1}{16}\int \sin^2\theta\cos\theta\,d\theta.$$

Now making the change of variables $u = \sin\theta$, $du = \cos\theta\,d\theta$,

$$\frac{1}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{16}\int u^2\,du = \frac{1}{16}\frac{u^3}{3} + C = \frac{1}{48}\sin^3\theta + C.$$

By the right triangle with sides $(A,O,H) = (4,\sqrt{x^2-16},x)$, $\sin\theta = \sqrt{x^2-16}/x$, so putting it all together

$$\int \frac{\sqrt{x^2-16}}{x^4}\,dx = \frac{(x^2-16)^{3/2}}{48x^3} + C.$$

Check by differentiating.

(b) It's probably easier to make an algebraic substition here; try it. However, the trig substitution also works. Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int \frac{3\sin\theta}{\sqrt{9-9\sin^2\theta}}3\cos\theta\,d\theta = \int 3\sin\theta\,d\theta = -3\cos\theta + C.$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$ or the appropriate right triangle,

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -3\sqrt{1-\sin^2\theta} + C = -3\sqrt{1-(x/3)^2} + C = -\sqrt{9-x^2} + C.$$

Check by differentiating. The definite integral is now

$$\int_0^{\sqrt{5}} \frac{x}{\sqrt{9-x^2}}\,dx = \sqrt{9} - \sqrt{9-5} = 3 - 2 = 1.$$

(c) Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$. Then

$$\int \frac{x^2}{(a^2-x^2)^{5/2}}\,dx = \int \frac{a^2\sin^2\theta}{a^5\cos^5\theta}a\cos\theta\,d\theta = \frac{1}{a^2}\int \tan^2\theta\sec^2\theta\,d\theta.$$

Now making the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$,

$$\frac{1}{a^2}\int \tan^2\theta\sec^2\theta\,d\theta = \frac{1}{a^2}\int u^2\,du = \frac{1}{a^2}\frac{u^3}{3} + C = \frac{\tan^3\theta}{3a^2} + C.$$

By using a right triangle with the appropriate sides,

$$\int \frac{x^2}{(a^2-x^2)^{5/2}}\,dx = \frac{1}{3a^2}\frac{x^3}{(a^2-x^2)^{3/2}} + C.$$

Check by differentiating.

(d) Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$. Then

$$\int \frac{x^2}{(a^2+x^2)^{5/2}}\, dx = \int \frac{a^2\tan^2\theta}{a^5\sec^5\theta} a\sec^2\theta\, d\theta = \frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta.$$

Since the power of cos is odd, make the substitution $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\frac{1}{a^2}\int \sin^2\theta\cos\theta\, d\theta = \frac{1}{a^2}\int u^2\, du = \frac{1}{a^2}\frac{u^3}{3} + C$$

Reversing the substitutions using the appropriate right triangle,

$$\int \frac{x^2}{(a^2+x^2)^{5/2}}\, dx = \frac{1}{a^2}\frac{\sin^3\theta}{3} + C = \frac{1}{3a^2}\frac{x^3}{(a^2+x^2)^{3/2}} + C.$$

Check by differentiating.

2. (a) Let $4x = 3\sec\theta$, $4dx = 3\sec\theta\tan\theta\, d\theta$. Then

$$\int \frac{dx}{\sqrt{16x^2-9}} = \frac{3}{4}\int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{4}\int \sec\theta d\theta = \frac{1}{4}\ln|\sec\theta+\tan\theta| + C.$$

Reversing the substitution, $\tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{16x^2-9}/3$ and

$$\int \frac{dx}{\sqrt{16x^2-9}} = \frac{1}{4}\ln|4x/3 - \sqrt{16x^2-9}/3| + C = \frac{1}{4}\ln|4x - \sqrt{16x^2-9}| + C$$

where $-(1/4)\ln 3$ has been absorbed into C. Check by differentiating.

(b) Here the algebraic substitution may work better than a trig substitution. Let $u = 4-9x^2$, $du = -18x\, dx$. Then

$$\int x\sqrt{4-9x^2}\, dx = \int u^{1/2}\frac{du}{-18} = -\frac{1}{18}\frac{u^{3/2}}{3/2} + C = -\frac{1}{27}(4-9x^2)^{3/2} + C.$$

(Check by differentiating.) Evaluating the definite integral,

$$\int_0^{2/3} x\sqrt{4-9x^2}\, dx = \frac{1}{27}(4-9(0)^2)^{3/2} - \frac{1}{27}\left(4-9\left(\frac{2}{3}\right)^2\right)^{3/2} = \frac{8}{27}.$$

To four decimal places I get the answer to the question is 0.2963.

(c) Make the trig substitution $2x = \tan\theta$, $2dx = \sec^2\theta\, d\theta$, we have

$$\int \sqrt{4x^2+1}\, dx = \int \sqrt{\sec^2\theta}\sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta.$$

By example 8 on page 523 of the textbook,

$$\int \sqrt{4x^2+1}\, dx = \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C = x\sqrt{4x^2+1} + \frac{1}{2}\ln|\sqrt{4x^2+1}+2x| + C.$$

Check by differentiating. Now the definite integral is

$$\int_0^1 \sqrt{4x^2+1}\, dx = 1\sqrt{5} + \frac{1}{2}\ln|\sqrt{5}+2| - 0 - \frac{1}{2}\ln|\sqrt{1}+0| = \sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5}).$$

(d) Let $ax = b\sec\theta$, $a\,dx = b\sec\theta\tan\theta\,d\theta$. Then

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = \frac{b}{a}\int \frac{\sec\theta\tan\theta\,d\theta}{b^5\tan^5\theta} = \frac{1}{ab^4}\int \frac{\cos^3\theta}{\sin^4\theta}\,d\theta.$$

It's possible to do the integral by changing the quotient of sines and cosines to a product of cotangent and cosecant, but since we are avoiding the use of those functions if possible, let's try another way. Change $\cos^2$ in the numerator to a function involving sines and make the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$:

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = \frac{1}{ab^4}\int \frac{(1-\sin^2\theta)\cos\theta}{\sin^4\theta}\,d\theta = \frac{1}{ab^4}\int u^{-4}-u^{-2}\,du = \frac{1}{ab^4}\left(\frac{u^3}{-3}-\frac{u^{-1}}{-1}\right)+C.$$

Reversing the substitutions using a $(A,O,H) = (b,(a^2x^2-b^2)^{1/2},ax)$ right triangle,

$$\int \frac{dx}{(a^2x^2-b^2)^{5/2}} = \frac{1}{ab^4}\left(-\frac{\csc^3\theta}{3}+\csc\theta\right)+C = -\frac{1}{ab^4}\left(\frac{a^3x^3}{3(a^2x^2-b^2)^{3/2}}-\frac{ax}{(a^2x^2-b^2)^{1/2}}\right)+C.$$

Check by differentiating.

3. These problems require completing the square.

(a) Write $x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$. Then making the substitution $x+1 = \tan\theta$, $dx = \sec^2\theta\,d\theta$,

$$\int \frac{dx}{(x^2+2x+2)^4} = \int \frac{\sec^2\theta\,d\theta}{\sec^8\theta} = \int \cos^6\theta\,d\theta.$$

By repeated application of reduction formula 74 at the back of the textbook,

$$\int \frac{dx}{(x^2+2x+2)^4} = \frac{1}{6}\sin\theta\cos^5\theta+\frac{5}{24}\sin\theta\cos^3\theta+\frac{15}{48}\sin\theta\cos\theta+\frac{15}{48}\theta+C.$$

Reversing the substitution by a $(1,x+1,\sqrt{x^2+2x+2})$ right triangle, $\sin\theta = (x+1)/\sqrt{x^2+2x+2}$, $\cos\theta = 1/\sqrt{x^2+2x+2}$, and

$$\int \frac{dx}{(x^2+2x+2)^4} = \frac{1}{6}\frac{x+1}{(x^2+2x+2)^3}+\frac{5}{24}\frac{x+1}{(x^2+2x+2)^2}+\frac{15}{48}\frac{x+1}{x^2+2x+2}+\frac{15}{48}\tan^{-1}(x+1)+C.$$

(b) Completing the square, $t^2-6t+1 = t^2-6t+9-8 = (t-3)^2-8$. Making the substitution $t-3 = 2\sqrt{2}\sec\theta$, $dt = 2\sqrt{2}\sec\theta\tan\theta\,d\theta$,

$$\int \frac{3\,dt}{\sqrt{t^2-6t+1}} = \int \frac{6\sqrt{2}\sec\theta\tan\theta\,d\theta}{2\sqrt{2}\tan\theta} = \int 3\sec\theta\,d\theta = 3\ln|\sec\theta+\tan\theta|+C.$$

Reversing the substitution, $\sec\theta = (t-3)/(2\sqrt{2})$, $\tan\theta = \sqrt{(t-3)^2/8-1} = \sqrt{t^2-6t+1}/(2\sqrt{2})$,

$$\int \frac{3\,dt}{\sqrt{t^2-6t+1}} = 3\ln\left|\frac{t-3}{2\sqrt{2}}+\frac{\sqrt{t^2-6t+1}}{2\sqrt{2}}\right|+C = 3\ln|t-3+\sqrt{t^2-6t+1}|+C$$

by the law of logarithms $\ln|a/b| = \ln a-\ln b$, absorbing the constant $-\ln 2\sqrt{2}$ into C. (Check by differentiating.)

(c) As with the previous problem, complete the square to obtain $t^2-6t+1 = (t-3)^2-8$, but this time make the algebraic substitution $u = (t-3)^2-8$, $du = 2(t-3)\,dt$:

$$\int \frac{t-3}{\sqrt{t^2-6t+1}}\,dt = \int \frac{du}{2u^{1/2}} = \frac{1}{2}\frac{u^{1/2}}{1/2}+C = \sqrt{t^2-6t+1}+C.$$

Check by differentiating.

(d) Adding the previous two results,

$$\int \frac{t}{\sqrt{t^2-6t+1}}\,dt = \int \frac{t-3}{\sqrt{t^2-6t+1}}\,dt + \int \frac{3}{\sqrt{t^2-6t+1}}\,dt = \sqrt{t^2-6t+1} + 3\ln|t-3+\sqrt{t^2-6t+1}| + C.$$

As usual, check by differentiating.

4. (a) Since the degree of the denominator is not greater than the degree of the numerator, we must first simplify the integrand by polynomial long division:

$$\frac{r^2}{r+4} = \frac{r(r+4)-4r}{r+4} = r - \frac{4r}{r+4} = r - \frac{4(r+4)-16}{r+4} = r - 4 + \frac{16}{r+4}.$$

Now the integral can be evaluated directly:

$$\int \frac{r^2}{r+4} = \int r - 4 + \frac{16}{r+4}\,dr = \frac{r^2}{2} - 4r + 16\ln|r+4| + C.$$

Check by differentiating.

(b) Again, the degree of the denominator is not greater than the degree of the numerator, so we must apply long division:

$$\frac{x^3-4x-10}{x^2-x-6} = \frac{x(x^2-x+6)+x^2-10x-10}{x^2-x+6} = x + \frac{1(x^2-x+6)-9x-16}{=} x + 1 - \frac{9x+16}{x^2-x+6}.$$

Check by evaluating $(x+1)(x^2-x+6)-(9x+16)$. Now factor the denominator $x^2-x+6=(x-3)(x+2)$ and apply partial fractions to obtain

$$\frac{9x+16}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}.$$

Equating the coefficients of similar powers of x in the numerators, we obtain the system

$$\begin{aligned} A+B &= 9 \\ 2A-3B &= 16 \end{aligned}$$

Adding 3 times the first equation to the second we get $A=43/5$; adding -2 times the first equation to the second we get $B=2/5$. You should check those results by substituting into the above equations. Altogether, the integrand is

$$\frac{x^3-4x-10}{x^2-x-6} = x + 1 - \frac{43/5}{x-3} - \frac{2/5}{x+2}$$

which you should check by placing the terms on the left hand side over a common denominator. It follows that the indefinite integral is

$$\int \frac{x^3-4x-10}{x^2-x-6}\,dx = \int x + 1 - \frac{43/5}{x-3} - \frac{2/5}{x+2}\,dx = \frac{x^2}{2} + x - \frac{43}{5}\ln|x-3| - \frac{2}{5}\ln|x+2| + C$$

(which you should check by integration) so the definite integral is

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6} = \frac{1}{2} + 1 - \frac{43}{5}\ln 2 - \frac{2}{5}\ln 3 + \frac{43}{5}\ln 3 + \frac{2}{5}\ln 2 = \frac{3}{2} - \frac{41}{5}\ln 2 + \frac{41}{5}\ln 3 = \frac{3}{2} + \frac{41}{5}\ln\frac{3}{2}.$$

Note the importance of the absolute value signs in the indefinite integral.

(c) The partial fractions decomposition of the integrand is

$$\frac{1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} = \frac{As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2}{s^2(s-1)^2}.$$

Expanding the numerator,

$$(A+C)s^3 + (-2A+B-C+D)s^2 + (A-2B)s + B = 0s^3 + 0s^2 + 0s + 1$$

from which we obtain $B = 1$, $A = 2$, $C = -2$, and $D = 1$. (Check.) Therefore our integral is

$$\int \frac{ds}{s^2(s-1)^2} = \int \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} ds = 2\ln s - \frac{1}{s} + 2\ln|s-1| - \frac{1}{s-1}.$$

As usual, check by differentiating.

(d) Again, long division gives

$$\frac{x^3}{x^3+1} = \frac{x^3+1-1}{x^3+1} = 1 - \frac{1}{x^3+1}.$$

To get any further we need to know how to factor x^3+1. Guessing a root, we have $(-1)^3 + 1 = 0$ so we know it has a factor of the form $x+1$. By long division we have

$$x^3 + 1 = (x+1)(x^2 - x + 1)$$

and the latter factor is irreducible because the discriminant $b^2 - 4ac = (-1)^2 - 4(1)(1) = -3$ is less than 0. So the partial fractions decomposition of the second term is

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{(A+B)x^2 + (-A+B+C)x + (A+C)}{x^2-x+1}$$

giving the system

$$\begin{aligned} A+B \quad\;\; &= 0 \\ -A+B+C &= 0 \\ A \quad\;\; -C &= 1 \end{aligned}$$

Adding the second and third equations above gives $B = 1$ from which we obtain $A = -1$ and $C = -2$. (Check.) Altogether our integral is

$$\int \frac{x^3}{x^3+1}\,dx = \int 1 - \frac{1}{x^3+1}\,dx = x + \int \frac{1}{x+1} - \frac{x+2}{x^2-x+1}\,dx = x + \ln|x+1| - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{5}{2}\int \frac{5}{x^2-x+1}\,dx.$$

The first integral on the right hand side above can be handled by the substitution $u = x^2 - x + 1$, $du = (2x-1)\,dx$. The second integral must be handled by completing the square to obtain

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

so the appropriate substitution is

$$x - \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta, \qquad dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta.$$

Carrying this through,

$$\int \frac{1}{x^2-x+1}\,dx = \int \frac{1}{(3/4)\tan^2\theta + 3/4}\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta = \frac{2}{\sqrt{3}}\int d\theta = \frac{2}{\sqrt{3}}\theta = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)\right)$$

Putting it all together,

$$\int \frac{x^3}{x^3+1}\,dx = x + \ln|x+1| - \frac{1}{2}\ln|x^2-x+1| - \frac{10}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)\right)$$

Guess what you should do at this point.

5. (a) Let $u = \sqrt{x+2}$, $du = (1/2)(x+2)^{-1/2}\,dx$, $2u\,du = dx$. Then the integral becomes

$$\int \frac{1}{1-\sqrt{x+2}}\,dx = \int \frac{2u}{1-u}\,du = \int \frac{-2(1-u)+2}{1-u}\,du = -2u - 2\ln|1-u| + C = -2\sqrt{x+2} - 2\ln|1-\sqrt{x+2}| + C.$$

Check by differentiating.

(b) Let $u = \sqrt[3]{x}$. Then $3u^2\,du = dx$ and

$$\int \frac{1}{1+\sqrt[3]{x}}\,dx = \int \frac{3u^2}{1+u}\,du = \int 3u - 3 + \frac{3}{1+u}\,du = \frac{3}{2}u^2 - 3u + 3\ln|1+u| + C = \frac{3}{2}x^{2/3} - 3x^{1/3} + 3\ln|1+x^{1/3}| + C.$$

Check by differentiating.

(c) Let $u = \sin x$, $du = \cos x\,dx$. Then the integral becomes

$$\int \frac{\cos x}{\sin^2 x + \sin x}\,dx = \int \frac{1}{u^2+u}\,du = \int \frac{1}{1+u} - \frac{1}{u}\,du = \ln|1+u| - \ln|u| + C = \ln|1+\sin x| - \ln|\sin x| + C.$$

Check by differentiating.

(d) This is a tricky one. Try the substitution $u = \sqrt[12]{x}$, $12u^1 1\,du = dx$, $\sqrt[3]{x} = u^4$, $\sqrt[4]{x} = u^3$:

$$\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx = \int \frac{12u^1 1}{u^4+u^3}\,du = 12\int \frac{u^8}{u+1}\,du.$$

The long division is rather tedious but is not difficult:

$$\frac{u^8}{u+1} = u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1}$$

(check) so the integral is

$$\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx = \frac{12}{8}u^8 - \frac{12}{7}u^7 + \frac{12}{6}u^6 - \frac{12}{5}u^5 + \frac{12}{4}u^4 - \frac{12}{3}u^3 + \frac{12}{2}u^2 - \frac{12}{1}u + \ln|u+1| + C.$$

Reversing the substitution gives

$$\int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\,dx = \frac{12}{8}x^{8/12} - \frac{12}{7}x^{7/12} + \frac{12}{6}x^{6/12} - \frac{12}{5}x^{5/12} + \frac{12}{4}x^{4/12} - \frac{12}{3}x^{3/12} + \frac{12}{2}x^{2/12} - \frac{12}{1}x^{1/12} + \ln|x^{1/12}+1| + C$$

Isn't that wonderful? Check by differentiating.

6. Since $t = \tan(x/2)$ we have

$$t^2 + 1 = \sec^2(x/2) \implies \cos^2(x/2) = \frac{1}{t^2+1}.$$

By the double angle formulas we have

$$\cos x = 2\cos^2(x/2) - 1 = \frac{2}{t^2+1} - 1 = \frac{1-t^2}{1+t^2}.$$

We also have

$$\sin^2 x = 1 - \cos^2 x = \frac{(1+t^2)^2}{(1+t^2)^2} - \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{4t^2}{(1+t^2)^2)}$$

which implies that

$$\sin x = \frac{2t}{1+t^2}$$

(at least for positive values of t; what could we do with negative values of t?). Differentiating,

$$dt = \frac{1}{2}\sec^2(x/2)\,dx \implies dx = \frac{2}{1+t^2}\,dt.$$

Armed with the above, we can integrate any rational trig function. If another method is available, we should use the other method, but in some cases the half-angle substitution is the only method available. For example,

$$\int \frac{dx}{3-5\sin x} = \int \frac{1}{3-10t/(1+t^2)}\frac{2\,dt}{1+t^2} = \int \frac{2}{3t^2-10t+3}\,dt.$$

Factoring the denominator,

$$3t^2-10t+3 = (3t-1)(t-3)$$

so

$$\frac{2}{3t^2-10t+3} = \frac{A}{3t-1} + \frac{B}{t-3} = \frac{(A+3B)t+(-3A-B)}{(3t-1)(t-3)}$$

from which we obtain $B = 1/4$, $A = -3/4$, and the integral is

$$\int \frac{dx}{3-5\sin x} = \int \frac{-3/4}{3t-1} + \frac{1/4}{t-3}\,dt = -\frac{1}{4}\ln|3t-1| + \frac{1}{4}\ln|t-3| + C = -\frac{1}{4}\ln|3\tan(x/2)-1| + \frac{1}{4}\ln|\tan(x/2)-3| + C.$$

Check by differentiating.