

MATH111-002 200630 Problem Set 6 Solutions DRAFT

Edward Doolittle

November 9, 2006

1. (a) By the trig substitution $x = \sqrt{2} \tan \theta$ we have $dx = \sqrt{2} \sec^2 \theta d\theta$, $x^2 + 2 = 2 \tan^2 \theta + 2 = 2 \sec^2 \theta$, so

$$\int \frac{1}{(x^2 + 2)^2} dx = \int \frac{1}{2 \sec^4 \theta} \sqrt{2} \sec^2 \theta d\theta = \frac{1}{\sqrt{2}} \int \cos^2 \theta d\theta.$$

By the double angle formula,

$$\int \frac{1}{(x^2 + 2)^2} dx = \frac{1}{\sqrt{2}} \int \frac{\cos 2\theta + 1}{2} d\theta = \frac{1}{2\sqrt{2}} \left(\frac{\sin 2\theta}{2} + \theta \right) + C.$$

Reversing the substitution,

$$\cos \theta = \sqrt{\frac{2}{x^2 + 2}}, \quad \sin \theta = \sqrt{\frac{x^2}{x^2 + 2}}, \quad \theta = \tan^{-1} \frac{x}{\sqrt{2}}$$

and so

$$\int \frac{1}{(x^2 + 2)^2} dx = \frac{1}{2\sqrt{2}} (\sin \theta \cos \theta + \theta) + C = \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}x}{x^2 + 2} + \tan^{-1} \frac{x}{\sqrt{2}} \right) + C.$$

Check by differentiating.

- (b) Let $u = -x^3$, $du = -3x^2 dx$, $x^2 dx = du/(-3)$. Then

$$\int x^2 e^{-x^3} dx = \int e^u \frac{du}{-3} = -\frac{1}{3} e^u + C = -\frac{1}{3} e^{-x^3} + C.$$

Check by differentiating.

- (c) Integrate by parts with $u = r$, $du = dr$, $dv = e^{r/3} dr$, and $v = 3e^{r/3}$, we have

$$\int r e^{r/3} dr = 3r e^{r/3} - \int 3e^{r/3} dr = 3r e^{r/3} - 9e^{r/3} + C.$$

Check by differentiating.

- (d) First make the substitution $u = \ln x$, $du = dx/x$:

$$\int \frac{\ln x}{x^3} dx = \int \frac{\ln x}{x^2} \frac{dx}{x} = \int \frac{u}{e^{2u}} du.$$

We can put that in more familiar form by writing

$$\int \frac{\ln x}{x^3} dx = \int u e^{-2u} du$$

which we know how to integrate by parts:

$$\int \frac{\ln x}{x^3} dx = -\frac{1}{2} u e^{-2u} - \frac{1}{4} e^{-2u} + C.$$

Reversing the substitution,

$$\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$$

Check by differentiating.

2. (a) Using the result of question 1(a), we have

$$\int_0^{\infty} \frac{1}{(x^2+2)^2} dx = \lim_{M \rightarrow \infty} \int_0^M \frac{1}{(x^2+2)^2} dx = \lim_{M \rightarrow \infty} \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}x}{x^2+2} + \tan^{-1} \frac{x}{\sqrt{2}} \right) \Big|_0^M = \frac{1}{2\sqrt{2}} \frac{\pi}{2}.$$

Therefore the improper integral converges to the above value.

- (b) Using the result of question 1(b), we have

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \lim_{M \rightarrow -\infty} \int_M^0 x^2 e^{-x^3} dx + \lim_{M \rightarrow \infty} \int_0^M x^2 e^{-x^3} dx = \lim_{M \rightarrow -\infty} -\frac{1}{3} e^{-M^3} + \frac{1}{3} + \lim_{M \rightarrow \infty} -\frac{1}{3} e^{-M^3} + \frac{1}{3}.$$

However, the first limit on the right hand side of the above does not exist (if M is large and negative, then $-M^3$ is very large and positive, and the exponential is even larger). Therefore the improper integral diverges. (It does not matter what happens to the second limit; if we know that one of them diverges, then the whole thing diverges.)

- (c) Using the result of question 1(c), we have

$$\int_{-\infty}^0 r e^{r/3} dr = \lim_{M \rightarrow -\infty} \int_M^0 r e^{r/3} dr = \lim_{M \rightarrow -\infty} 3r e^{r/3} - 9e^{r/3} \Big|_M^0 = -9$$

by L'Hôpital's rule. Therefore the improper integral converges to the above value.

- (d) Using the result of question 1(d), we have

$$\int_1^{\infty} \frac{\ln x}{x^3} dx = \lim_{M \rightarrow \infty} -\frac{\ln x}{2x^2} - \frac{1}{4x^2} \Big|_1^M = \frac{1}{4}$$

by L'Hôpital's rule. Therefore the given improper integral converges to the above value.

3. We first integrate the corresponding indefinite integral. Then we identify points in the domain of integration for which the integrand diverges; such points are sometimes called "poles". Then we write the improper integral as a limit as the endpoint(s) of integration approach the poles. Finally, we evaluate the limit.

- (a) The integrand has a pole at $x = 2$ so the integral is type II improper and should be interpreted as

$$\int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{a \rightarrow 2^-} \int_0^a \frac{1}{\sqrt{4-x^2}} dx.$$

In order to evaluate the definite integral above, we first try to evaluate the indefinite integral

$$\int \frac{1}{\sqrt{4-x^2}} dx.$$

Make the trig substitution $x = 2 \sin \theta$, $dx = 2 \cos \theta d\theta$ to obtain

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{2 \cos \theta}{2 \cos \theta} d\theta = \theta + C = \sin^{-1} \frac{x}{2} + C.$$

Check by differentiation. The definite integral then becomes

$$\int_0^a \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1} \frac{a}{2}$$

so the improper integral becomes

$$\int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{a \rightarrow 2^-} \int_0^a \frac{1}{\sqrt{4-x^2}} dx = \lim_{a \rightarrow 2^-} \sin^{-1} \frac{a}{2} = \frac{\pi}{2},$$

so the given integral converges to that value.

- (b) The integrand has a pole at $z = 2$ so the integral should be interpreted as

$$\int_0^3 (z-2)^{-1/3} dz = \lim_{a \rightarrow 2^-} \int_0^a (z-2)^{-1/3} dz + \lim_{b \rightarrow 2^+} \int_b^3 (z-2)^{-1/3} dz.$$

The next step is to find the indefinite integral

$$\int (z-2)^{-1/3} dz$$

which is easy with the substitution $u = z - 2$, $du = dz$:

$$\int (z-2)^{-1/3} dz = \int u^{-1/3} du = \frac{u^{2/3}}{2/3} + C = \frac{3u^{2/3}}{2} + C = \frac{3(z-2)^{2/3}}{2} + C.$$

The definite integrals are then

$$\begin{aligned} \int_0^a (z-2)^{-1/3} dz &= \frac{3(a-2)^{2/3}}{2} - \frac{3(-2)^{2/3}}{2} = \frac{3}{2}(a-2)^{2/3} - \frac{3}{2}(4)^{1/3} \\ \int_b^3 (z-2)^{-1/3} dz &= \frac{3(1)^{2/3}}{2} - \frac{3(b-2)^{2/3}}{2} = \frac{3}{2} - \frac{3}{2}(b-2)^{2/3} \end{aligned}$$

so the improper integral is

$$\int_0^3 (z-2)^{-1/3} dz = \lim_{a \rightarrow 2^-} \frac{3}{2}(a-2)^{2/3} - \frac{3}{2}4^{1/3} + \lim_{b \rightarrow 2^+} \frac{3}{2} - \frac{3}{2}(b-2)^{2/3} = \frac{3}{2} - \frac{3}{2}4^{1/3}.$$

To four decimal points, I have the value is -0.8811 .

- (c) We first search for poles of the integrand in the domain of integration. The integrand becomes infinite when its denominator goes to zero, so we look for roots of the equation $x^2 + x - 6 = 0$. The polynomial factors and the equation becomes $(x+3)(x-2) = 0$ with roots $x = -3$, $x = 2$. The integrand does have a pole in the domain of integration, namely $x = 2$, so the integral is type II improper and must be interpreted as

$$\int_0^4 \frac{1}{x^2+x-6} dx = \lim_{a \rightarrow 2^-} \int_0^a \frac{1}{x^2+x-6} dx + \lim_{b \rightarrow 2^+} \int_b^4 \frac{1}{x^2+x-6} dx.$$

To evaluate the indefinite integral, we use partial fractions.

$$\int \frac{1}{x^2+x-6} dx = \frac{1}{5} \left(\int \frac{1}{x-2} - \frac{1}{x+3} \right) dx = \frac{1}{5} \ln|x-2| - \frac{1}{5} \ln|x+3| + C.$$

So the definite integrals are

$$\begin{aligned} \int_0^a \frac{1}{x^2+x-6} dx &= \frac{1}{5} \ln|a-2| - \frac{1}{5} \ln|a+3| - \frac{1}{5} \ln 2 + \frac{1}{5} \ln 3 \\ \int_b^4 \frac{1}{x^2+x-6} dx &= \frac{1}{5} \ln 2 - \frac{1}{5} \ln 7 - \frac{1}{5} \ln|b-2| + \frac{1}{5} \ln|b+3|. \end{aligned}$$

However, $\lim_{a \rightarrow 2^-} \ln|a-2| = -\infty$, so it follows that the improper integral is divergent.

- (d) As $x \rightarrow 0^+$ the integrand diverges, so the integral is improper of type I. To evaluate the indefinite integral, make the substitution $u = \ln x$, $du = dx/x$, $dx = e^u du$ to obtain

$$\int \frac{\ln x}{\sqrt{x}} dx = \int \frac{u}{e^{u/2}} e^u du = \int u e^{u/2} du = 2u e^{u/2} - 4e^{u/2} + C = 2\sqrt{x} \ln x - 4\sqrt{x} + C.$$

The improper integral is then

$$\int_0^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} 2\sqrt{a} \ln a - 4\sqrt{a} - 2\sqrt{1} \ln 1 + 4\sqrt{1} = \lim_{a \rightarrow 0^+} 2\sqrt{a} \ln a - 0 - 0 + 4.$$

To evaluate the limit on the right hand side of the above, we use L'Hôpital's rule:

$$\lim_{a \rightarrow 0^+} 2\sqrt{a} \ln a = \lim_{a \rightarrow 0^+} \frac{2 \ln a}{a^{-1/2}} = \lim_{a \rightarrow 0^+} \frac{2}{a(-1/2)a^{-3/2}} = \lim_{a \rightarrow 0^+} -4a^{1/2} = 0$$

so the improper integral converges to the value 4.

4. (a) We may be able to evaluate the indefinite integral in this case, but there is no need. We just estimate the integrand. The integrand looks a lot like $2/x$, and the integral of $2/x$ diverges on the interval $[1, \infty)$, so we suspect that the given improper integral diverges, so we estimate the integrand from below. We have

$$\frac{2}{x} < \frac{2 + e^{-x}}{x}$$

because $e^{-x} > 0$. Since

$$\int_1^{\infty} \frac{2}{x} dx = \lim_{M \rightarrow \infty} 2 \ln M = \infty$$

we conclude that the given integral also diverges by the comparison test.

- (b) The denominator looks a lot like $\sqrt{x^6} = x^3$, and the integral of $x/x^3 = 1/x^2$ converges on the interval $[1, \infty)$, so we suspect that the given integral converges. So we try to estimate the integrand from above. In fact

$$\frac{x}{\sqrt{1+x^6}} \leq \frac{x}{x^3} = \frac{1}{x^2}$$

on $[1, \infty)$ because

$$\sqrt{x^6} < \sqrt{1+x^6}.$$

Since

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{M \rightarrow \infty} 1 - \frac{1}{M}$$

converges, it follows from the comparison test that the given integral converges.

- (c) On the domain of integration $[0, 1]$, the function e^{-x} is (very roughly speaking) a constant, so the properties of the integrand e^{-x}/\sqrt{x} should be (very roughly speaking) the same as the properties of the integrand $1/\sqrt{x}$. The integral

$$\int_0^1 \frac{1}{\sqrt{x}} dx$$

converges by direct evaluation as in question 3 (quite similar to 3(b), in fact), so we expect that the given integral should converge too. To confirm our expectation, we use the comparison test. On the interval $[0, 1]$ we have $0 < e^{-x} \leq 1$ (draw a graph to see why), so on the interval $(0, 1]$ we have

$$0 < \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}.$$

Since the integral of the right hand side of the above converges on $(0, 1]$, the given integral also converges by the comparison test.

- (d) The integrand diverges as $x \rightarrow 0^+$ so the integral is improper of type II. Near 0 the sine function is approximately the same as x , so the integral is roughly similar to

$$\int_0^{\pi/2} \frac{1}{x^2} dx$$

which diverges. Therefore we expect that our integral diverges, so we try to estimate the integrand from below. We would like a better estimate for $\sin x$ on the interval $[0, \pi/2]$. We can in fact say that $\sin x \leq x$ on that entire interval, not just near 0: in order to prove that inequality you could show that it holds at 0 and then show that $x - \sin x$ is an increasing function on the interval. In any case, we have

$$0 < \frac{1}{x} \leq \frac{1}{\sin x} \implies 0 < \frac{1}{x^2} \leq \frac{1}{x \sin x}.$$

Since the integral over $(0, \pi/2]$ of the middle term of the above inequality diverges, it follows by the comparison test that the given integral diverges.

We actually did more than was required. We could have solved the problem using only the “zeroth order estimate” for \sin , i.e., $0 \leq \sin x \leq 1$ on $[0, \pi/2]$ instead of the first order estimate $0 \leq \sin x \leq x$ on $[0, \pi/2]$. (Try it!) However, sometimes the first order estimate is required. Consider, for example, whether the improper integral

$$\int_0^{\pi/2} \frac{dx}{\sqrt{x} \sin x}$$

converges or diverges.

5. Taking the limit throughout the given formula,

$$\lim_{t \rightarrow \infty} \int_0^t x^n e^{ax} dx = \lim_{t \rightarrow \infty} \frac{1}{a} t^n e^{at} - \frac{1}{a} 0^n e^{a \cdot 0} - \frac{n}{a} \lim_{t \rightarrow \infty} \int_0^t x^{n-1} e^{ax} dx;$$

the second limit above is 0 by the hint so the formula becomes

$$\int_0^{\infty} x^n e^{ax} dx = -\frac{n}{a} \int_0^{\infty} x^{n-1} e^{ax} dx$$

which is actually simpler than the corresponding formula on a finite interval. It follows that

$$\int_0^{\infty} x^2 e^{-3x} dx = \frac{2}{3} \int_0^{\infty} x^1 e^{-3x} dx = \frac{2}{3} \frac{1}{3} \int_0^{\infty} x^0 e^{-3x} dx = \frac{2}{9} \frac{1}{-3} e^{-3x} \Big|_0^{\infty} = \frac{2}{27}.$$

(Question: this improper integral can give us a generalization of the factorial function to any nonnegative real number; how?)

6. The integral is type I improper, so to make sense of it we must write

$$\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \lim_{M \rightarrow \infty} \int_0^M \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx.$$

The latter integral can be evaluated as follows: for the first term in the integrand, let $u = x^2 + 1$ giving $du = 2x dx$, $du/2 = x dx$,

$$\int \frac{x}{x^2+1} dx = \int \frac{1}{u} \frac{du}{2} = \frac{1}{2} \ln|u| + C_1 = \frac{1}{2} \ln|x^2+1| + C_1.$$

(Check by differentiating.) For the second term in the integrand we have

$$\int \frac{C}{3x+1} dx = \frac{C}{3} \ln|3x+1| + C_2.$$

(Check by differentiating.) Altogether we have

$$\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \lim_{M \rightarrow \infty} \frac{1}{2} \ln|M^2+1| - \frac{C}{3} \ln|3M+1|.$$

We need to find a value of C for which the latter limit converges. We turn it into a quotient by exponentiating so we can apply L'Hôpital's rule or something similar:

$$\lim_{M \rightarrow \infty} \frac{1}{2} \ln|M^2 + 1| - \frac{C}{3} \ln|3M + 1| = \ln \left(\lim_{M \rightarrow \infty} e^{(1/2) \ln|M^2 + 1| - (C/3) \ln|3M + 1|} \right) = \ln \left(\lim_{M \rightarrow \infty} \frac{(M^2 + 1)^{1/2}}{(3M + 1)^{C/3}} \right).$$

Actually, we don't have to apply L'Hôpital's rule to evaluate the latter limit. Just divide the numerator and denominator through by M to obtain

$$\lim_{M \rightarrow \infty} \frac{1}{2} \ln|M^2 + 1| - \frac{C}{3} \ln|3M + 1| = \ln \left(\lim_{M \rightarrow \infty} \frac{(1 + 1/M^2)^{1/2}}{(3M^{1-3/C} + M^{-3/C})^{C/3}} \right).$$

For the limit to exist, we need $\lim_{M \rightarrow \infty} 3M^{1-3/C} + M^{-3/C} > 0$ which implies $1 - 3/C \geq 0$. For the logarithm of the limit to exist, we need $\lim_{M \rightarrow \infty} 3M^{1-3/C} + M^{-3/C} < \infty$ which implies $1 - 3/C \leq 0$. We conclude that we must have $1 - 3/C = 0$, i.e., $C = 3$, in which case

$$\lim_{M \rightarrow \infty} \frac{1}{2} \ln|M^2 + 1| - \frac{C}{3} \ln|3M + 1| = \ln \left(\lim_{M \rightarrow \infty} \frac{(M^2 + 1)^{1/2}}{(3M + 1)} \right) = \ln \left(\lim_{M \rightarrow \infty} \frac{(1 + (1/M)^2)^{1/2}}{(3 + 1/M)} \right) = \ln \frac{1}{3}.$$

In conclusion, the given improper integral converges if and only if $C = 3$, in which case the value of the improper integral is $-\ln 3$.