

# MATH 111 Problem Set 8 Solutions DRAFT

Edward Doolittle

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1. (a) Taking the derivative, we have

$$y' = x - \frac{1}{4x}.$$

Squaring, we have

$$(y')^2 = x^2 - \frac{1}{2} + \frac{1}{16x^2}.$$

It follows that

$$1 + (y')^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

so

$$L = \int_2^4 \sqrt{\left(x + \frac{1}{4x}\right)^2} dx = \int_2^4 x + \frac{1}{4x} dx = \left. \frac{x^2}{2} + \frac{\ln x}{4} \right|_2^4 = 8 + \frac{1}{4} \ln 4 - 2 - \frac{1}{4} \ln 2 = 6 + \frac{1}{4} \ln 2.$$

- (b) **The hard way.** We have

$$(y')^2 = \left(\frac{1}{2}(4x)^{-1/2} \cdot 4\right)^2 = x^{-1}$$

so

$$L = \int_0^2 \sqrt{1 + \frac{1}{x}} dx.$$

Now we make the rationalizing substitution  $u = \sqrt{1 + 1/x}$ ,  $du = 1/2(1 + 1/x)^{-1/2} \cdot -1/x^2 dx$ ,  $-2udu/(u - 1)^2 = dx$  to obtain

$$\int \sqrt{1 + \frac{1}{x}} dx = \int u \cdot \frac{-2u}{(u-1)^2} du = \int \frac{-2u^2}{u^2 - 2u + 1} du = \int -2 + \frac{-4u + 2}{(u-1)^2} du.$$

We continue with the partial fractions decomposition, etc.

**The easy way.** We integrate with respect to  $y$  instead of  $x$ . We have

$$\frac{dx}{dy} = \frac{y}{2}$$

so

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4}$$

so

$$\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int \sqrt{1 + \frac{y^2}{4}} dy.$$

Making the substitution  $y = (1/2) \tan \theta$ ,  $dy = (1/2) \sec^2 \theta d\theta$  we have

$$\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int \sqrt{1 + \tan^2 \theta} \frac{1}{2} \sec^2 \theta d\theta = \frac{1}{2} \int \sec^3 \theta d\theta.$$

We then look up the integral for  $\sec^3 \theta$  in a table and proceed from there.

(c) We have

$$y' = \frac{d}{dx} \ln(e^x + 1) - \frac{d}{dx} \ln(e^x - 1) = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{e^{2x} - e^x - e^{2x} - e^x}{(e^x + 1)(e^x - 1)} = \frac{-2e^x}{e^{2x} - 1}.$$

Squaring,

$$(y')^2 = \frac{4e^{2x}}{(e^{2x} - 1)^2} = \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1},$$

so

$$\sqrt{1 + (y')^2} = \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{e^{4x} - 2e^{2x} + 1}} = \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{e^{4x} - 2e^{2x} + 1}} = \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}} = \frac{e^{2x} + 1}{e^{2x} - 1}$$

(assuming  $x > 0$ ; if  $x < 0$ , a slight change to the final equality is required; what do you think that might be?). It follows that the arc length is

$$L = \int_a^b \frac{e^{2x} + 1}{e^{2x} - 1} dx.$$

To do the indefinite integral, let  $u = e^{2x} - 1$ ,  $du = 2e^{2x} dx$ ,  $du/(2u + 2) = dx$ ,

$$\int \frac{e^{2x} + 1}{e^{2x} - 1} dx = \int \frac{u + 2}{u} \frac{du}{2u + 2} = \int \frac{1}{u} - \frac{1}{2u + 2} du = \ln|u| - \frac{1}{2} \ln|2u + 2| + C = \ln(e^{2x} - 1) - \ln(e^x) + C.$$

Therefore the arc length is

$$L = \ln(e^x - e^{-x}) \Big|_a^b = \ln\left(\frac{e^b - e^{-b}}{e^a - e^{-a}}\right).$$

(d) We have  $y' = (1/\cos x)(-\sin x) = -\tan x$ , so

$$L = \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\pi/3} \sec x dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3} = \ln(\sec(\pi/3) + \tan(\pi/3)).$$

You can calculate the value using a calculator or you can use a 30-60-90 degree right triangle to find the exact value.

2. The arc length function is

$$s(x) = \int_1^x \sqrt{1 + (y')^2} dx.$$

(Strictly speaking, we should use a dummy variable inside the integral so that the two different meanings of the letter  $x$  don't get confused.) Filling in the details,

$$1 + (y')^2 = 1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4} = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2$$

so

$$s(x) = \int_1^x \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} dx = \int_1^x x^2 + \frac{1}{4x^2} dx = \left. \frac{x^3}{3} - \frac{1}{4x} \right|_1^x = \frac{x^3}{3} - \frac{1}{4x} - \frac{1}{3} + \frac{1}{4}.$$

3. We have  $y' = 1 + 1/x^2$  so the left hand side is

$$xy' + y = x + \frac{1}{x} + x - \frac{1}{x} = 2x$$

which agrees with the right hand side.

4. First we verify that the function is a solution to the differential equation. The left hand side is

$$y' - (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \cos x - \cos x) = \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x$$

which agrees with the right hand side, so the function is a solution to the differential equation.

Second, we verify that the given function satisfies the initial condition:

$$y(0) = \sin(0) \cos(0) - \cos(0) = 0(1) - 1 = -1$$

so the function also satisfies the initial condition  $y(0) = -1$ , so the function is a solution to the initial value problem.

5. (a) If  $y = e^t$  then  $y' = e^t$ ,  $y'' = e^t$  and  $y'' + 2y' + y = 4e^t \neq 0$  so  $e^t$  is not a solution.  
(b) If  $y = e^{-t}$  then  $y' = -e^{-t}$ ,  $y'' = e^{-t}$  and  $y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0$  so  $e^{-t}$  is a solution.  
(c) If  $y = te^{-t}$  then  $y' = e^{-t} - te^{-t}$ ,  $y'' = -e^{-t} - e^{-t} + te^{-t} = -2e^{-t} + te^{-t}$ , and

$$y'' + 2y' + y = -2e^{-t} + te^{-t} + 2e^{-t} - 2te^{-t} + te^{-t} = 0$$

so  $te^{-t}$  is a solution.

- (d) If  $y = t^2 e^{-t}$  then  $y' = 2te^{-t} - t^2 e^{-t}$ ,  $y'' = 2e^{-t} - 2te^{-t} - 2te^{-t} + t^2 e^{-t} = 2e^{-t} - 4te^{-t} + t^2 e^{-t}$  and

$$y'' + 2y' + y = 2e^{-t} - 4te^{-t} + t^2 e^{-t} + 4te^{-t} - 2t^2 e^{-t} + t^2 e^{-t} = 2e^{-t} \neq 0$$

so  $t^2 e^{-t}$  is not a solution.

6. We have  $y' = re^{rt}$ ,  $y'' = r^2 e^{rt}$  so

$$y'' + y' - 6y = r^2 e^{rt} + re^{rt} - 6e^{rt} = (r^2 + r - 6)e^{rt}.$$

The above expression is zero if and only if  $r^2 + r - 6 = 0$ , i.e., if  $r = -3$  or  $r = 2$ .

7. We did this one in class.

8. (a) The left hand side of the equation is

$$y' = -\frac{1}{2}(c-x^2)^{-3/2} \cdot -2x = x(c-x^2)^{-3/2}$$

by the chain rule. The right hand side of the equation is

$$xy^3 = x\left((c-x^2)^{-1/2}\right)^3 = x(c-x^2)^{-3/2}.$$

The two sides agree, so the given function is a solution to the differential equation.

(b) We try to find a value of  $c$  so that the function  $y = (c - x^2)^{-1/2}$  has the property  $y(0) = 2$ . In terms of  $c$  we have  $y(0) = (c - 0^2)^{-1/2} = c^{-1/2}$ . If  $y(0) = 2$  we have  $c^{-1/2} = 2$  which implies  $c^{1/2} = 1/2$  which implies  $c = (1/2)^2 = 1/4$ . Checking, the function  $y = ((1/4) - x^2)^{-1/2}$  is a solution to the given initial value problem.

(c) You can graph members of the family of functions, or try letting  $c \rightarrow \infty$ , or try solving the separable differential equation  $dy/dx = xy^3$ , and you'll see that there is an "envelope solution"  $y = 0$  which is not of the form  $y = (c - x^2)^{-1/2}$ .

9. Differentiating implicitly, we have

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \implies y' = -\frac{y^{1/3}}{x^{1/3}} \implies (y')^2 = \frac{y^{2/3}}{x^{2/3}} = \frac{1 - x^{2/3}}{x^{2/3}} = x^{-2/3} - 1.$$

Integrating over the branch of the curve in quadrant I we have

$$L = \int_0^1 \sqrt{1 + (y')^2} dx = \int_0^1 \sqrt{x^{-2/3}} dx = \int_0^1 x^{-1/3} dx = \left. \frac{3}{2}x^{2/3} \right|_0^1 = \frac{3}{2}.$$

Since there are four branches each with the same length by symmetry, we have the total length of the curve is  $4 \cdot (3/2) = 6$ .

10. We have  $y' = (4/3)x^{1/3}$ ,  $(y')^2 = (16/9)x^{2/3}$ , so

$$L = \int_0^1 \sqrt{1 + \frac{16}{9}x^{2/3}} dx.$$

This is a tough integral, but not impossible. Let  $u = x^{1/3}$ . Then  $du = \frac{1}{3}x^{-2/3} dx$ ,  $dx = 3u^2 du$ ,

$$\int \sqrt{1 + \frac{16}{9}x^{2/3}} dx = \int \sqrt{1 + \frac{16}{9}u^2} 3u^2 du.$$

Now make the trig substitution  $(4/3)u = \tan \theta$ ,  $du = (3/4) \sec^2 \theta d\theta$ ,

$$\int \sqrt{1 + \frac{16}{9}x^{2/3}} dx = \int \sqrt{1 + \tan^2 \theta} \frac{81}{64} \tan^2 \theta \sec^2 \theta d\theta = \frac{81}{64} \int \tan^2 \theta \sec^3 \theta d\theta.$$

Probably the best way to tackle the above integral is to change all the tans to secs using the Pythagorean identity and then use a reduction formula (or integration by parts) for the integral of powers of sec. The integral can be evaluated in closed form so the arc length can be calculated exactly. (If anyone has completed the calculation, I'd appreciate it if I could borrow your notes to complete this solution set.)