MATH 111 Problem Set 9 Solutions DRAFT

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1. (a) Separating xs and ys,

$$y\,dy = \frac{dx}{x}$$
$$\int y\,dy = \int \frac{dx}{x}$$
$$\frac{1}{2}y^2 = \ln|x| + C.$$

If we can solve for y, we should, and we obtain

$$y^2 = \ln x^2 + 2C$$
$$y = (\ln x^2 + C)^{1/2}$$

where I used the logarithm law $2\ln a = \ln a^2$ and replaced $2C$ with another constant C' which I then changed into a C. (Both of those operations are optional, but make it easier to check.) You should check that the above family of functions actually is a solution.

(b) This is in the notes.

(c) Separating variables,

$$\frac{dy}{y^2} = \sin x\,dx$$
$$-y^{-1} = -\cos x + C$$
$$y^{-1} = \cos x - C$$
$$y = (\cos x - C)^{-1}.$$

Check.

(d) Separating variables,

$$\frac{2\ln u}{u}\,du = t\,dt$$
$$\int \frac{2\ln u}{u}\,du = \int t\,dt.$$

The integral on the left side can be done with the substitution $v = \ln u$, $dv = du/u$ to obtain

$$\int 2v\,dv = \frac{1}{2}t^2 + C$$
$$v^2 = \frac{1}{2}t^2 + C(\ln u)^2 \qquad = \frac{1}{2}t^2 + C$$

where the last step reverses the substitution $v = \ln u$. Now, solving for u explicitly, we have

$$\ln u = \left(\frac{1}{2}t^2 + C\right)^{1/2}$$
$$u = e^{(t^2/2+C)^{1/2}}.$$

Check.

2. The strategy for the initial value problems is first to find a family of functions which satisfies the differential equation, and then to find a member of the family which also satisfies the initial condition, which we do by solving for C. Note that all the differential equations in this question are separable.

   (a) Separating variables,

$$\begin{aligned}
\frac{1+y^2}{y}\,dy &= \cos x\,dx \\
\int \frac{1+y^2}{y}\,dy &= \int \cos x\,dx \\
\int \frac{1}{y} + y\,dy &= \sin x + C \\
\ln|y| + \frac{1}{2}y^2 &= \sin x + C.
\end{aligned}$$

   It is not clear how to solve for y explicitly, so we just leave it as is. Now we find the member of that family of implicit functions which satisfies the initial condition $y(0) = 1$ by substituting $x = 0$ and $y = 1$ and then solving for C:

$$\ln 1 + \frac{1}{2}(1)^2 = \sin 0 + C \frac{1}{2} \qquad = C.$$

   Therefore the solution to the initial value problem is the function $y(x)$ satisfying

$$\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}.$$

   You should check that it actually solves the initial value problem. You will need to use implicit differentiation.

   (b) Separating variables,

$$\begin{aligned}
P^{-1/2}\,dP &= t^{1/2}\,dt \\
2P^{1/2} &= \frac{2}{3}t^{3/2} + C \\
P^{1/2} &= \frac{1}{3}t^{3/2} + C_1 \\
P &= \left(\frac{1}{3}t^{3/2} + C\right)^2.
\end{aligned}$$

   For the initial condition we have $P = 2$ when $t = 1$ so

$$\begin{aligned}
2 &= \left(\frac{1}{3}1^{3/2} + C\right)^2 \\
\sqrt{2} &= \frac{1}{3} + C \\
C &= \sqrt{2} - \frac{1}{3}.
\end{aligned}$$

   The solution to the initial value problem is therefore

$$P = \left(\frac{1}{3}t^{3/2} - \frac{1}{3} + \sqrt{2}\right)^2.$$

   Check.

(c) Separating variables we have

$$\begin{aligned}
e^{-y}\,dy &= t\,dt \\
-e^{-y} &= \frac{1}{2}t^2 + C \\
e^{-y} &= -\frac{1}{2}t^2 - C \\
-y &= \ln(-t^2/2 - C) \\
y &= -\ln(-t^2/2 - C).
\end{aligned}$$

The initial condition says that $y = 0$ when $t = 1$ so we have

$$\begin{aligned}
0 &= -\ln(-1/2 - C) \\
0 &= \ln(-1/2 - C) \\
1 &= -1/2 - C \\
C &= -3/2.
\end{aligned}$$

Therefore the solution to the initial value problem is

$$y = -\ln\left(\frac{3}{2} - \frac{t^2}{2}\right).$$

Check.

(d) We first move the y from the left to the right and then separate ys and xs:

$$\begin{aligned}
x\frac{dy}{dx} + y &= y^2 \\
x\frac{dy}{dx} &= y^2 - y \\
\frac{dy}{y(y-1)} &= \frac{dx}{x} \\
\int \frac{1}{y-1} - \frac{1}{y}\,dy &= \int \frac{dx}{x} \\
\ln|y-1| - \ln|y| &= \ln|x| + C.
\end{aligned}$$

We can actually solve for y explicitly. By a law of logarithms,

$$\begin{aligned}
\ln|(y-1)/y| &= \ln|x| + C \\
|(y-1)/y| &= C_1|x| \\
1 - \frac{1}{y} &= Cx \\
\frac{1}{y} &= 1 - Cx \\
y &= \frac{1}{1 - Cx}.
\end{aligned}$$

At this point, it might be a good idea to check whether we have the correct family of functions. The next step is to pick a value of C so that the function satisfies the initial condition $y = -1$ when $x = 1$, i.e.,

$$\begin{aligned}
-1 &= \frac{1}{1 - C(1)} \\
C - 1 &= 1 \\
C &= 2.
\end{aligned}$$

Therefore the solution to the initial value problem is

$$y = \frac{1}{1-2x}.$$

Now you definitely should check that the above function actually is a solution to the initial value problem.

3. For each of these equations, we put it into the standard form $y' + P(x)y = Q(x)$, then find an integrating factor, multiply the equation in standard form by the factor, use the product rule in reverse, then integrate.

   (a) We put the equation into the standard form

   $$y' - 5y = x$$

   from which we see that $P(x) = -5$. Therefore an integrating factor is $I(x) = e^{-5x}$. Multiplying the equation in standard form by the integrating factor we obtain

   $$e^{-5x}y' - 5e^{-5x}y = xe^{-5x}.$$

   The left hand side can be written as $(e^{-5x}y)'$ so we obtain the equation

   $$(e^{-5x}y)' = xe^{-5x}.$$

   Integrating both sides,

   $$e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

   where the latter result was obtained by integration by parts. Solving for y,

   $$y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}.$$

   You should check that the above familiy of functions actually satisfies the equation.

   (b) Write the equation in standard form $y' + P(x)y = Q(x)$:

   $$y' + \frac{2}{x}y = \frac{1}{x^2}\cos^2 x.$$

   An integrating factor is

   $$I(x) = e^{\int P(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln x} = x^2.$$

   The equation was originally in a form which could have been integrated directly; if you noticed that, you could have gone directly to the next step. The equation is

   $$(x^2y)' = \cos^2 x \implies x^2y = \int \cos^2 x\,dx = \frac{1}{2}\int \cos(2x) + 1\,dx = \frac{1}{4}\sin(2x) + \frac{1}{2}x + C.$$

   Therefore the general solution can be written

   $$y = \frac{1}{4x^2}\sin(2x) + \frac{1}{2x} + C\frac{1}{x^2}.$$

   You should check that the above family of functions actually solve the equation.

   (c) In standard form we have

   $$y' - \tan x\, y = x\sin 2x.$$

   An integrating factor is

   $$I(x) = e^{\int -\tan x\,dx} = e^{\ln|cosx|} = \cos x,$$

where we can drop the absolute value signs because of the restriction on x: $-\pi/2 < x < \pi/2$ implies that $|\cos x| = \cos x$. Anyway, multiplying by the integrating factor we have

$$\cos x\, y' - \sin x y = x \sin 2x\, \cos x \implies (\cos x y)' = x \sin 2x\, \cos x \implies \cos x y = \int x \sin 2x\, \cos x\, dx.$$

The integral on the right hand side is difficult but not impossible for us. The best way to handle it is to use the angle addition formulas

$$\sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$$
$$\sin(2x - x) = \sin 2x \cos x - \cos 2x \sin x.$$

Adding the above formulas gives

$$\sin(3x) + \sin x = 2 \sin 2x \cos x$$

so the integral can be written

$$\frac{1}{2}\int x \sin 3x + x \sin x\, dx = -\frac{1}{6} x \cos 3x + \frac{1}{18} \sin 3x - \frac{1}{2} x \cos x + \frac{1}{2} \sin x + C,$$

integrating by parts. The solution to the differential equation is therefore

$$y = \frac{1}{\cos x}\left(-\frac{1}{6} x \cos 3x + \frac{1}{18} \sin 3x - \frac{1}{2} x \cos x + \frac{1}{2} \sin x + C\right).$$

It may be possible to simplify with trig identities. You should check that the solution satisfies the differential equation.

(d) In standard form the equation is

$$r' + \frac{1}{t \ln t} r = \frac{1}{\ln t} e^t.$$

To find an integrating factor we need to integrate

$$\int \frac{1}{t \ln t}\, dt = \int \frac{1}{u}\, du = \ln u + C = \ln \ln t + C$$

where the substitution $u = \ln t$ was applied. Therefore an integrating factor is

$$I(x) = e^{\int P(x)\, dx} = e^{\ln \ln t} = \ln t.$$

Multiplying by the integrating factor gives

$$\ln t\, r' + \frac{1}{t} r = e^t \implies (\ln t\, r)' = e^t \implies \ln t\, r = \int e^t\, dt = e^t + C \implies r = \frac{e^t}{\ln t} + \frac{C}{\ln t}.$$

Check.

4. As in question 2, we solve the differential equation and then pick a value of C for which the solution satisfies the initial condition.

(a) By 3(a), the solution to the differential equation is

$$y = -\frac{1}{5} x - \frac{1}{25} + C e^{5x}.$$

For the initial condition, we have $y = 1$ and $x = 0$, i.e.

$$1 = -\frac{1}{5}(0) - \frac{1}{25} + C e^{5(0)} \implies 1 = -\frac{1}{25} + C \implies C = \frac{26}{25}.$$

So the solution to the initial value problem is

$$y = -\frac{1}{5} x - \frac{1}{25} + \frac{26}{25} e^{5x}.$$

You should double check that solution.

(b) Skipping a whole bunch of steps, the equation with the integrating factor is

$$t^2y' + 2ty = t^4 \implies (t^2y)' = t^4 \implies t^2y = \frac{1}{5}t^5 + C \implies y = \frac{1}{5}t^3 + Ct^{-2}.$$

Using the inital condition, we have

$$0 = \frac{1}{5}(1)^3 + C(1)^{-2} \implies C = -\frac{1}{5}$$

so the solution to the initial value problem is

$$y = \frac{1}{5}t^3 - \frac{1}{5}t^{-2}.$$

Check.

(c) With the appropriate integrating factor the equation becomes

$$x^{1/2}y' + \frac{1}{2}x^{-1/2}y = 3x^{1/2} \implies (x^{1/2}y)' = 3x^{1/2} \implies x^{1/2}y = 2x^{3/2} + C \implies y = 2x + Cx^{-1/2}.$$

Using the initial condition we have

$$20 = 2(4) + C(4)^{-1/2} \implies 12 = C\frac{1}{2} \implies C = 24,$$

so the solution to the initial value problem is

$$y = 2x + 24x^{-1/2}.$$

Check.

(d) Here the integrating factor is not obvious. We write the equation in standard form

$$y' - \frac{1}{x(x+1)}y = 1.$$

To find an integrating factor we must integrate

$$\int -\frac{1}{x(x+1)}\,dx = \int \frac{1}{x+1} - \frac{1}{x}\,dx = \ln|x+1| - \ln|x| = \ln\left|\frac{x+1}{x}\right|,$$

so an integrating factor is

$$I(x) = e^{\ln|(x+1)/x|} = \frac{x+1}{x}$$

where the absolute value signs were dropped because of the condition $x > 0$. Multiplying by the integrating factor we obtain the equation

$$\frac{x+1}{x}y' - \frac{1}{x^2}y = \frac{x+1}{x} \implies \left(\frac{x+1}{x}y\right)' = 1 + \frac{1}{x} \implies \frac{x+1}{x}y = x + \ln x + C \implies y = \frac{x^2}{x+1} + \frac{x\ln x}{x+1} + C\frac{x}{x+1}.$$

The initial condition gives

$$0 = \frac{1}{2} + \frac{0}{2} + C\frac{1}{2} \implies C = -1,$$

so the solution to the initial value problem is

$$y = \frac{x^2}{x+1} + \frac{x\ln x}{x+1} - \frac{x}{x+1}.$$

Check.

5. (a)
   (b)
   (c)
   (d)
   (e)
6. (a)
   (b)
   (c)
   (d)