MATH 111 Problem Set 10 Solutions DRAFT

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Please check these results carefully and let me know if there are any errors. I haven't had the time to do a careful check of the work myself.

1. For each of these problems, we double check that it is homogeneous and then solve by making the substitution $y = xv$.

   (a) The right hand side can be written $-1-(y/x)$ which is a function of y/x, so the equation is homogeneous. We have

   $$\begin{aligned} \frac{d}{dx}(xv) &= -\frac{x+xv}{x} \\ x\frac{dv}{dx}+v &= -1-v \\ x\frac{dv}{dx} &= -1-2v \\ \frac{1}{1+2v}\,dv &= -\frac{1}{x}\,dx \end{aligned}$$

   which is homogeneous. Integrating both sides,

   $$\frac{1}{2}\ln|1+2v| = -\ln|x|+C \implies \ln|1+2v| = -2\ln|x|+C_1 \implies v = \frac{1}{2}(C_2x^{-2}-1) \implies y = \frac{1}{2}(Cx^{-1}-x).$$

   Check.

   (b) We can check that the equation is homogeneous in a way similar to the previous or by

   $$-\frac{(ky)^2+(ky)(kx)}{(kx)^2} = -\frac{k^2(y^2+yx)}{k^2x^2} = -\frac{y^2+yx}{x^2}$$

   so the equation is homogeneous. Substituting $y = xv$,

   $$x\frac{dv}{dx}+v = -\frac{(xv)^2+(xv)x}{x^2} = -v^2-v \implies x\frac{dv}{dx} = -v^2-2v \implies \frac{1}{v(v+2)}\,dv = -\frac{1}{x}\,dx$$

   which is homogeneous. Integrating,

   $$\frac{1}{2}\ln|v|-\frac{1}{2}\ln|v+2| = -\ln|x|+C \implies \frac{v}{v+2} = C_1x^{-2} \implies \frac{v+2}{v} = C_2x^2 \implies \frac{2}{v} = Cx^2-1 \implies y = \frac{2x}{Cx^2-1}.$$

   Check.

   (c) It's straightforward to verify that the equation is homogeneous. Substituting $y = xv$,

   $$x\frac{dv}{dx}+v = \frac{x+3xv}{3x+xv} = \frac{1+3v}{3+v} \implies x\frac{dv}{dx} = \frac{1+3v-v(3+v)}{3+v} = \frac{1-v^2}{3+v} \implies \frac{3+v}{(v-1)(v+1)}\,dv = -\frac{1}{x}\,dx.$$

   Integrating both sides,

   $$\int \frac{2}{v-1}-\frac{1}{v+1}\,dv = \int -\frac{1}{x}\,dx \implies 2\ln|v-1|-\ln|v+1| = -\ln|x|+C \implies \frac{v^2-2v+1}{v+1} = C_1x^{-1}.$$

It's possible to solve explicitly for v, e.g.,

$$\frac{(v+1)^2-4(v+1)+3}{v+1}=Cx^{-1} \implies v+1+3(v+1)^{-1}=Cx^{-1}+4 \implies (v+1)^2-(Cx^{-1}+4)(v+1)+3=0$$

and use the quadratic equation, but it's unnecessary. We can leave the solution in implicit form; just remember to reverse the substitution, so the solution in implicit form is

$$\frac{y^2-2xy+x^2}{xy+x^2}=Cx^{-1}.$$

Check by implicit differentiation.

(d) The equation is homogeneous because

$$\frac{(ky)+\sqrt{(kx)^2-(ky)^2}}{kx}=\frac{ky+\sqrt{k^2}\sqrt{x^2-y^2}}{kx}=\frac{k(y+\sqrt{x^2-y^2}}{kx}=\frac{y+\sqrt{x^2-y^2}}{x}.$$

Making the substitution $y=xv$,

$$x^2\frac{dv}{dx}+xv=xv+\sqrt{x^2-(xv)^2} \implies x\frac{dv}{dx}=\sqrt{1-v^2} \implies \frac{1}{\sqrt{1-v^2}}\,dv=-\frac{1}{x}\,dx.$$

Integrating both sides,

$$\sin^{-1}v=-\ln|x|+C \implies v=\sin(C-\ln|x|) \implies y=x\sin(C-\ln|x|).$$

Check.

2. We check that the equation is homogeneous, then solve by making the substitution $y=xv$ and solving the resulting separable equation, then pick a value of C for which the corresponding member of the family of solutions also satisfies the initial condition.

(a) Both functions on the left and right sides are cubic in x,y so the equation is homogeneous. Substituting $y=xv$,

$$x^3v^2\frac{dv}{dx}+x^3v^3=x^3v^3-x^3 \implies v^2\frac{dv}{dx}=-1 \implies v^2\,dv=-dx \implies \frac{1}{3}v^3=-x+C \implies y=x(C-3x)^{1/3}.$$

You should probably check that family of solutions. The initial condition says $y=2$ when $x=1$ so

$$2=1(C-3(1))^{1/3} \implies 8=C-3 \implies C=11$$

so the solution to the initial value problem is $y=x(11-3x)^{1/3}$. You should definitely check that that answer satisfies the conditions of the initial value problem.

(b)

(c)

(d)

3. We write each of the equations in the form $M(x,y)\,dx+N(x,y)\,dy=0$ and test whether $\partial M/\partial y=\partial N/\partial x$.

(a) Here $M(x,y)=2x-1$ and $N(x,y)=3y+7$ so we have $\partial M/\partial y=0$, $\partial N/\partial x=0$, the two partials are equal, so the equation is exact.

(b) Here we have $M(x,y)=2x+y$, $N(x,y)=-(x+6y)$ (note that the negative sign *must* go with N), $\partial M/\partial y=1$, $\partial N/\partial x=-1$, so the equation is not exact.

(c) We have $M(x,y)=x^3+y^3$, $N(x,y)=3xy^2$, $\partial M/\partial y=3y^2$, $\partial N/\partial x=3y^2$. The two partials are equal so the equation is exact.

(d) The equation in standard form is $(1+\ln x+y/x)dx+(\ln y-1)dy=0$ so we have $M(x,y)=1+\ln x+y/x$, $N(x,y)=\ln y-1$, $\partial M/\partial y=1/x$, $\partial N/\partial x=0$, so the equation is not exact.

4. We should verify that each of the equations is exact and then solve by partial integration.

   (a) Here $M(x,y)=5x+4y$, $N(x,y)=4x-8y^2$, $\partial M/\partial y=4$, $\partial N/\partial x=4$, so the equation is exact. We have

   $$\frac{\partial F}{\partial x}=M(x,y)=5x+4y$$

   so by partial integration in the x variable, the most general form for F is

   $$F(x,y)=\int M(x,y)\,dx=\frac{5}{2}x^2+4xy+g(y)$$

   where g is an arbitrary function of y. Differentiating partially with respect to y we have

   $$N(x,y)=\frac{\partial F}{\partial y}=4x+g'(y) \implies 4x-8y^3=4x+g'(y) \implies g'(y)=-8y^3 \implies g(y)=-2y^4(+C).$$

   Altogether we have

   $$F(x,y)=\frac{5}{2}x^2+4xy-2y^4$$

   so the (implicit) solution to the differential equation is

   $$\frac{5}{2}x^2+4xy-2y^4=C.$$

   You should check that answer by implicit differentiation.

   (b)

   (c) This equation is both homogeneous and exact (check) so we could solve it either way. We will solve it as an exact equation, but you should try solving it as a homogeneous equation and compare solutions. By partial integration have

   $$\frac{\partial F}{\partial x}=M(x,y)=x^2-y^2 \implies F(x,y)=\int (x^2-y^2)\,dx=\frac{1}{3}x^3-xy^2+g(y).$$

   Differentiating partially with respect to y,

   $$N(x,y)=\frac{\partial F}{\partial y}=-2xy+g'(y) \implies y^2-2xy=-2xy+g'(y) \implies g'(y)=y^2 \implies g(y)=\frac{1}{3}y^3(+C).$$

   Therefore we have

   $$F(x,y)=\frac{1}{3}x^3-xy^2+\frac{1}{3}y^3(+C)$$

   and the solution to the equation is

   $$\frac{1}{3}x^3-xy^2+\frac{1}{3}y^3=C.$$

   It is too difficult to solve explicitly for y, so we leave the solution in implicit form. You should check by implicit differentiation.

   (d)

5. For each of these problems, we verify that the equation is exact, then solve by partial integration, then pick a member of the family of solutions that satisfies the initial condition.

(a) We have $\partial M/\partial y = 2(x+y)\partial(x+y)/\partial y = 2x+2y$ and $\partial N/\partial x = 2y+2x$, so the equation is exact. Partially integrating M with respect to x,

$$F(x,y) = \int M(x,y)\,dx = \int (x+y)^2\,dx = \frac{1}{3}(x+y)^3 + g(y)$$

where g is an arbitrary function of y. Partially differentiating with respect to y,

$$N(x,y) = \frac{\partial F}{\partial y} = (x+y)^2 + g'(y) \implies 2xy + x^2 - 1 = x^2 + 2xy + y^2 + g'(y) \implies g'(y) = -1 - y^2 \implies g(y) = -y - \frac{1}{3}y^3.$$

Altogether we have

$$F(x,y) = \frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3$$

so the general solution to the equation is

$$\frac{1}{3}(x+y)^3 - y - \frac{1}{3}y^3 = C.$$

At the initial point we have $x = 1$ and $y = 1$ so

$$\frac{1}{3}(1+1)^3 - 1 - \frac{1}{3}1^3 = C \implies \frac{2}{3} - 1 - \frac{1}{3} = C \implies C = -\frac{2}{3}$$

so the solution to the initial value problem (in implicit form) is

$$(x+y)^3 - 3y - y^3 = -2.$$

It is possible to solve explicitly for y, but difficult and unnecessary. You can check the solution by implicit differentiation.

(b)

(c)

(d)

6.