MATH 111 Sample Final Exam Solutions DRAFT

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1. (a) When $x = 0$ we have $y = f(0) = (e^0 + 1)/(0 + 1) = 2$ so the tangent line passes through the point $(0, 2)$. Furthermore, the derivative of f is

$$f'(x) = \frac{(x+1)e^x - (e^x+1)(1)}{(x+1)^2}$$

so the slope of the tangent line is $m = f'(0) = ((0+1)e^0 - (e^0+1)(1))/(0+1)^2 = (1-2)/1 = -1$. Therefore the point-slope equation of the tangent line is $y - 2 = (-1)(x - 0)$. Various simplifications are possible but not necessary.

(b) Note that $f(0) = 2$ so $f^{-1}(2) = 0$. We have

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{-1} = -1.$$

That solution is adequate; however, note that there is a second possible x such that $f(x) = 2$ because f is not one-to-one, so there is another answer to the question, but it's very hard to find.

2. (a) Substituting $x = 0$ to the expression under the limit we obtain $0/0$ so L'Hôpital's rule is applicable. Differentiating the numerator and denominator of the fraction, we have

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{6x^2} = \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{12x}.$$

Substituting $x = 0$ into the expression under the limit on the right we again obtain $0/0$ so L'Hôpital's rule is again applicable and we obtain

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{6x^2} = \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{12x} = \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{12} = \frac{4}{12} = \frac{1}{3}.$$

(b) This limit is of the form $0 \cdot \infty$ so we re-write

$$\lim_{x\to 0^+} x(\log_2 x)^2 = \lim_{x\to 0^+} \frac{(\log_2 x)^2}{1/x}$$

which is of the form ∞/∞ so L'Hôpital's rule applies. We have

$$\lim_{x\to 0^+} x(\log_2 x)^2 = \lim_{x\to 0^+} \frac{(\log_2 x)^2}{1/x} = \lim_{x\to 0^+} \frac{2(\log_2 x)(1/(\ln 2x))}{-(1/x)^2} = -\frac{2}{\ln 2} \lim_{x\to 0^+} \frac{\log_2 x}{(1/x)}$$

which is again of the form ∞/∞ so by L'Hôpital's rule

$$\lim_{x\to 0^+} x(\log_2 x)^2 = -\frac{2}{\ln 2} \lim_{x\to 0^+} \frac{\log_2 x}{(1/x)} = -\frac{2}{\ln 2} \lim_{x\to 0^+} \frac{1/(\ln 2x)}{-(1/x)^2} = \frac{2}{(\ln 2)^2} \lim_{x\to 0^+} x = 0.$$