MATH111-002 200630 Sample Midterm Test 1 Solutions DRAFT

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1. (a) Recall that if $y = e^u$ then $y' = e^u u'$, so we have
$$y' = e^{e^x} \cdot \frac{d}{dx} e^x = e^{e^x} e^x.$$

   (b) Before we take the derivative, some initial simplification is helpful:
$$y = \log_2(x^3) = 3\log_2(x) = 3\frac{\ln x}{\ln 2}$$
so
$$y' = \frac{3}{(\ln 2)x}.$$

   (c) By the chain rule and then the product rule,
$$y' = \frac{1}{2}(1 + xe^{-2x})^{-1/2}\frac{d}{dx}(1 + xe^{-2x}) = \frac{1}{2}(1 + xe^{-2x})^{-1/2}\left(e^{-2x} - 2xe^{-2x}\right).$$

2. (a) First, the indefinite integral is
$$\int e^{-3x}\,dx = \int e^u \frac{du}{-3} = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-3x} + C$$
(check by differentiating). Now the definite integral can be evaluated by the fundamental theorem of calculus:
$$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big|_0^5 = -\frac{1}{3}\left(e^{-15} - 1\right).$$

   (b) We can re-write this integral as
$$\int \frac{e^x + 1}{e^x}\,dx = \int \frac{e^x}{e^x} + \frac{1}{e^x}\,dx = \int 1 + e^{-x}\,dx = x - e^{-x} + C.$$

   (c) Make the substitution $u = x^2$, $du = 2x\,dx$ to obtain
$$\int x2^{x^2}\,dx = \frac{1}{2}\int 2^u\,du.$$
Now, by memorizing the formula or substituting $2 = e^{\ln 2}$ the integral can be evaluated:
$$\int x2^{x^2}\,dx = \frac{1}{2}\int e^{(\ln 2)u}\,du = \frac{1}{2}\frac{1}{\ln 2}e^{(\ln 2)u} + C = \frac{1}{2\ln 2}2^{x^2} + C.$$

3. (a) $f'(x) = 2 - \sin x$.

   (b) From the graph of $\sin x$ we have $-\sin x \geq -1$ so $f'(x) = 2 - \sin x \geq 2 - 1 = 1 > 0$.

   (c) Since $f'(x) > 0$ for all x, f is increasing, so it is one-to-one, so it is invertible.

(d) The best way to do this is to guess. $f(0) = 2(0) + \cos(0) = 0 + 1 = 1$, so $f^{-1}(1) = 0$.

(e) By our formula for the derivative of an inverse function,

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}.$$

4. (a) Taking the logarithm of both sides (assuming $y > 0$), we have

$$\ln y = 5\ln(2x+1) + 6\ln(x^4 - 3).$$

Note that we cannot simplify any further; in particular $\ln(2x+1)$ is NOT the same thing as $\ln(2x) + \ln(1)$. Now, differentiating the above formula we have

$$\frac{y'}{y} = 5\frac{1}{2x+1}\frac{d}{dx}(2x+1) + 6\frac{1}{x^4-3}\frac{d}{dx}(x^4-3) = \frac{10}{2x+1} + \frac{24x^3}{x^4-3}.$$

Multiplying through by $y = (2x+1)^5(x^4-3)^6$ we have

$$y' = (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right).$$

(b) Taking the logarithm of both sides (assuming $x, y > 0$),

$$\ln y = \ln x^{\sin x} = (\sin x)\ln x.$$

Differentiating (the RHS by the product rule),

$$\frac{y'}{y} = (\cos x)\ln x + (\sin x)\frac{1}{x}.$$

Multiplying through by $y = x^{\sin x}$,

$$y' = x^{\sin x}\left((\cos x)\ln x + (\sin x)\frac{1}{x}\right).$$

(c) Taking the logarithm of both sides (assuming $x, y > 0$),

$$\ln(x^y) = \ln(y^x) \implies y\ln x = x\ln y.$$

Differentiating implicitly and then solving for y',

$$\begin{aligned}
\frac{d}{dx}(y\ln x) &= \frac{d}{dx}(x\ln y) \\
y'\ln x + y\frac{1}{x} &= \ln y + x\frac{y'}{y} \\
y'\ln x - y'\frac{x}{y} &= \ln y - \frac{y}{x} \\
y' &= \frac{\ln y - (y/x)}{\ln x - (x/y)}.
\end{aligned}$$

5. Taking one derivative at a time,

$$\frac{d}{dx}(x^2\ln x) = 2x\ln x + x^2\frac{1}{x} = 2x\ln x + x$$

so

$$\frac{d^2}{dx^2}(x^2\ln x) = \frac{d}{dx}(2x\ln x + x) = 2\ln x + 2x\frac{1}{x} + 1 = 2\ln x + 3$$

so

$$\frac{d^3}{dx^3}(x^2\ln x) = \frac{d}{dx}(2\ln x + 3) = \frac{2}{x}.$$

6. This is based on the last problem of Problem Set 1. We need to show that the function $f(x) = e^x - (1+x)$ is always nonnegative when $x \geq 0$. Note that when $x = 0$, $f(x) = f(0) = e^0 - (1+0) = 0$ so the result is true then; now we would like to show that $f(x)$ is increasing for $x > 0$ which would imply that $f(x) > f(0) = 0$ for $x > 0$. To do so, we take the derivative: $f'(x) = e^x - 1 > 0$ for $x > 0$ by the graph of f (or by the argument in the solutions to Problem Set 1). Therefore $f(x)$ is increasing for $x > 0$, so $f(x) > 0$ for all $x > 0$, so $e^x \geq 1 + x$ for all $x \geq 0$.