MATH111-002 200630 Sample Midterm 2 Solutions DRAFT

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1. Integrating by parts with parts $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, $v = e^x$ gives

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx.$$

We must also do the latter integral by integration by parts, this time with the parts $u = 2x$, $dv = e^x\,dx$, $du = 2\,dx$, $v = e^x$ giving

$$\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + C.$$

Putting it all together,

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Check by differentiating.

2. First we do the indefinite integral

$$\int \sin^5 x\,dx = \int \sin^4 x\ \sin x\,dx = \int (1-\cos^2 x)^2\ \sin x\,dx = \int (1 - 2\cos^2 x + \cos^4 x)\ \sin x\,dx.$$

Making the substitution $u = \cos x$, $-du = \sin x\,dx$ gives

$$\int \sin^5 x\,dx = -\int (1 - 2u^2 + u^4)\,du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

You can check that result by differentiating and applying trig identities. Now evaluating the definite integral,

$$\int_0^{\pi/2} \sin^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\Big|_0^{\pi/2} = -0 + \frac{2}{3}0^3 - \frac{1}{5}0^5 - (-1) - \frac{2}{3}1^3 + \frac{1}{5}1^5 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$

3. By partial fractions,

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2} = \frac{(A+B)x + (-2A+5B)}{(x+5)(x-2)}$$

so we must have

$$\begin{aligned} A + B &= 1 \\ -2A + 5B &= -9. \end{aligned}$$

Adding 2 times the first equation to the second gives $B = -1$ from which it follows that $A = 2$. (Check.) So the integral becomes

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2}{x+5} - \frac{1}{x-2}\,dx = 2\ln|x+5| - \ln|x-2| + C.$$

Check by differentiating.

4. Complete the square under the square root sign to obtain

$$5+4x-x^2 = 9-(4-4x+x^2) = 9-(x-2)^2.$$

Making the substitution $x-2 = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$ gives

$$\int \sqrt{5+4x-x^2}\, dx = \int \sqrt{9-(x-2)^2}\, dx = \int \sqrt{9-9\sin^2\theta}\, 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta.$$

The integral of $\cos^2$ can be done most efficiently by using the double angle formula

$$\cos 2\theta = 2\cos^2\theta - 1 \implies \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

to obtain

$$\int \sqrt{5+4x-x^2}\, dx = 9\int \frac{\cos 2\theta}{2} + \frac{1}{2}\, d\theta = \frac{9}{4}\sin 2\theta + \frac{9}{2}\theta + C.$$

In order to reverse the substitution we need to apply the double angle formula for sin which gives

$$\int \sqrt{5+4x-x^2}\, dx = \frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + C = \frac{1}{2}(x-2)\sqrt{9-(x-2)^2} + \frac{9}{2}\sin^{-1}\frac{x-2}{3} + C.$$

Check by differentiating.

5. (a) We need to integrate by parts. Let $u = \sin^{n-1} x$, $dv = \sin x\, dx$, $du = (n-1)\sin^{n-2} x\, \cos x\, dx$, $v = -\cos x$. Then

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\, dx.$$

Applying the Pythagorean identity $\cos^2 x = 1 - \sin^2 x$,

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx.$$

Solving the above equation for $\sin^n x$,

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx,$$

or in other words,

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x\, \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx,$$

as required. (Question: for what values of n is the above analysis valid?)

(b) First, we will need a version of the reduction formula for definite integrals. Integration by parts for definite integrals says

$$\int_0^{\pi/2} \sin^n x\, dx = -\frac{1}{n}\cos x\, \sin^{n-1} x\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx,$$

i.e.,

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

because $\sin 0 = 0$ and $\cos(\pi/2) = 0$. Now, for those of you who are familiar with mathematical induction, the above provides the induction step for a proof of the result. If you're not familiar with mathematical induction, I'm satisfied if you just work out the answer for the first few values of n. For $n = 1$ we have

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}\int_0^{\pi/2} \sin^1 x\, dx = \frac{2}{3}\left(-\cos x\Big|_0^{\pi/2}\right) = \frac{2}{3};$$

for $n = 2$ we have

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \int_0^{\pi/2} \sin^5 x\, dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x\, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 5};$$

for $n = 3$ we have

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7} \int_0^{\pi/2} \sin^5 x\, dx = \frac{6}{7} \cdot \frac{2 \cdot 4}{3 \cdot 5} = \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7},$$

and so on. The general pattern continues as above.