MATH122 200610 Midterm Test 1C Solutions

Edward Doolittle

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1. (a) We are looking for solutions to the linear system $\left[\begin{array}{ccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \end{array}\right]$. In detail, the augmented matrix in question is

$$\left[\begin{array}{ccc|c} -3 & -5 & -1 & 5 \\ 1 & 3 & 3 & 1 \\ 2 & 9 & 12 & 8 \end{array}\right].$$

Swapping row 1 and row 2,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ -3 & -5 & -1 & 5 \\ 2 & 9 & 12 & 8 \end{array}\right].$$

Adding 3 times row 1 to row 2 and -2 times row 1 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 4 & 8 & 8 \\ 0 & 3 & 6 & 6 \end{array}\right].$$

Multiplying row 2 by 1/4 and row 3 by 1/3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right].$$

Adding -1 times row 2 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We're now in row echelon form, from which we can see that there is a solution. Continuing to reduced row echelon form, we add -3 times row 2 to row 1 to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This gives us the general solution $x_1 = -5 + 3s$, $x_2 = -2 - 2s$, $x_3 = s$. Choosing $s = 0$, for example, gives us a particular solution $x_1 = -5$, $x_2 = -2$, $x_3 = 0$. Then $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{b}$ for those values of the x's, so $\mathbf{b}$ is a linear combination of the $\mathbf{u}$'s. (Check.)

(b) In parametric form, the set of all x's that satisfy the equation is given by

$$\mathbf{x} = \begin{bmatrix} -5 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

so one possible geometric description of the solution set is the line through the point $(-5, -2, 0)$ parallel to the vector $(3, -2, 1)$.

(c) To answer this question we must figure out whether there is a non-trivial solution to the system $\left[\begin{array}{ccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{0} \end{array}\right]$. The calculations are exactly parallel to those of part (a), except that the final column is replaced with all 0's. In detail, the augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -5 & -1 & 0 \\ 1 & 3 & 3 & 0 \\ 2 & 9 & 12 & 0 \end{array}\right].$$

Swapping row 1 and row 2,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ -3 & -5 & -1 & 0 \\ 2 & 9 & 12 & 0 \end{array}\right].$$

Adding 3 times row 1 to row 2 and -2 times row 1 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right].$$

Multiplying row 2 by 1/4 and row 3 by 1/3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

Adding -1 times row 2 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We're now in row echelon form, and we can see that there is more than one solution to the system because there is a free variable, so the **u**'s are linearly dependent. (Find a non-trivial linear combination of the **u**'s that is **0**.)

We could answer this question without re-doing the row reduction; we can see from the row echelon form of the coefficient matrix in question 1(a) that there will be a free variable. It's enough to note that not every column of the coefficient matrix is a pivot column.

(d) To answer this question we must figure out whether there exists a solution to the system $\left[\begin{array}{ccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{c} \end{array}\right]$, where $\mathbf{c}$ is an arbitrary vector in $\mathbb{R}^3$. The calculations are exactly parallel to those of parts (a) and (c), except that the final column is replaced with all c's. In detail, the augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -5 & -1 & c_1 \\ 1 & 3 & 3 & c_2 \\ 2 & 9 & 12 & c_3 \end{array}\right].$$

Swapping row 1 and row 2,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & c_2 \\ -3 & -5 & -1 & c_1 \\ 2 & 9 & 12 & c_3 \end{array}\right].$$

Adding 3 times row 1 to row 2 and -2 times row 1 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & c_2 \\ 0 & 4 & 8 & c_1 + 3c_2 \\ 0 & 3 & 6 & c_3 - 2c_2 \end{array}\right].$$

Multiplying row 2 by 1/4 and row 3 by 1/3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & c_2 \\ 0 & 1 & 2 & (1/4)c_1 + (3/4)c_2 \\ 0 & 1 & 2 & (1/3)c_3 - (2/3)c_2 \end{array}\right].$$

Adding -1 times row 2 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & c_2 \\ 0 & 1 & 2 & (1/4)c_1 + (3/4)c_2 \\ 0 & 0 & 0 & -(1/4)c_1 - c_2 + (1/3)c_3 \end{array}\right].$$

We're now in row echelon form, and we can see that for some values of $\mathbf{c}$ (e.g., $c_1 = 0$, $c_2 = 0$, $c_3 = 1$) the system is inconsistent, so some $\mathbf{c}$'s cannot be realized as linear combinations of the **u**'s, i.e., the **u**'s do not span $\mathbb{R}^3$.

Again, we could have answered this question without re-doing the row reduction; we can see from the row echelon form of the coefficient matrix in question 1(a) that there will be an inconsistent equation. It's enough to note that not every row of the coefficient matrix is a pivot row.

2. (a) We need to show two things: $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and $T(c\mathbf{u}) = cT(\mathbf{u})$:

$$\begin{aligned} &T(\mathbf{u}+\mathbf{v}) \\ &= T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= ((u_1+v_1) - 2(u_2+v_2), \\ &\quad 3(u_2+v_2) - (u_3+v_3), \\ &\quad\quad 2(u_3+v_3) + (u_1+v_1)) \\ &= ((u_1 - 2u_2) + (v_1 - 2v_2), \\ &\quad (3u_2 - u_3) + (3v_2 - v_3), \\ &\quad\quad (2u_3 + u_1) + (2v_3 + v_1)) \\ &= (u_1 - 2u_2, 3u_2 - u_3, 2u_3 + u_1) \\ &\quad + (v_1 - 2v_2, 3v_2 - v_3, 2v_3 + v_1) \\ &= T(\mathbf{u}) + T(\mathbf{v}), \end{aligned}$$

and

$$\begin{aligned} &T(c\mathbf{u}) \\ &= T(cu_1, cu_2, cu_3) \\ &= ((cu_1) - 2(cu_2), 3(cu_2) - (cu_3), 2(cu_3) + (cu_1)) \\ &= c(u_1 - 2u_2, 3u_2 - u_3, 2u_3 + u_1) \\ &= cT(\mathbf{u}. \end{aligned}$$

(b) Evaluating T on the standard unit vectors, $T(1,0,0) = (1,0,1)$, $T(0,1,0) = (-2,3,0)$, and $T(0,0,1) = (0,-1,2)$, so the standard matrix for T is

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

3. (a) The matrix-vector product $A\mathbf{x}$ only makes sense when $\mathbf{x} \in \mathbb{R}^4$, so the domain of T is $\mathbb{R}^4$. The result of a matrix-vector product $A\mathbf{x}$ is a vector in $\mathbb{R}^3$ so the codomain of T is $\mathbb{R}^3$.

(b) The transformation T is onto if and only if the columns of A span the codomain. Since the property of spanning the codomain is preserved under row operations (see 1(d) above, for example), we put A into row echelon form. Adding -3 times row 1 to row 2, and -1 times row 1 to row 3, we have

$$\begin{bmatrix} 1 & 1 & 6 & 1 \\ 3 & 4 & 20 & 5 \\ 1 & -1 & k & -3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 6 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & k-6 & -4 \end{bmatrix}.$$

Adding 2 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & 3 & -10 & -6 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & k-2 & 0 \end{bmatrix}.$$

The columns of the above matrix span $\mathbb{R}^3$ if and only if each row contains a pivot element, which is true if and only if $k+8 \neq 0$. Therefore the mapping T is onto for all values of $k \neq 2$.

(c) The transformation T is one-to-one if and only if the columns of A are linearly independent. Since the property of linear independence of columns is preserved under row operations (see 1(c) above, for example), we put A into row echelon form. We've already done all the work in the previous answer. The columns are linearly independent if and only if there is a pivot element in each column. However, no matter what the value of k is, there cannot be a pivot element in the fourth column. We can conclude that there is no value of k for which T is one-to-one.

We didn't even need to use the row echelon form to answer this question, because we know that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

4. Either by solving two different linear system, or by guessing, we have

$$\begin{aligned} \mathbf{e}_1 &= 2(5\mathbf{e}_1 - 3\mathbf{e}_2) + 3(-3\mathbf{e}_1 + 2\mathbf{e}_2) \\ \mathbf{e}_2 &= 3(5\mathbf{e}_1 - 3\mathbf{e}_2) + 5(-3\mathbf{e}_1 + 2\mathbf{e}_2). \end{aligned}$$

Applying T and using linearity, we have

$$\begin{aligned} T(\mathbf{e}_1) &= T(2(5\mathbf{e}_1 - 3\mathbf{e}_2) + 3(-3\mathbf{e}_1 + 2\mathbf{e}_2)) \\ &= 2T(5\mathbf{e}_1 - 3\mathbf{e}_2) + 3T(-3\mathbf{e}_1 + 2\mathbf{e}_2) \\ &= 2(2\mathbf{e}_1 + 3\mathbf{e}_2) + 3(-\mathbf{e}_1 - 2\mathbf{e}_2) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{e}_2) &= T(3(5\mathbf{e}_1 - 3\mathbf{e}_2) + 5(-3\mathbf{e}_1 + 2\mathbf{e}_2)) \\ &= 3T(5\mathbf{e}_1 - 3\mathbf{e}_2) + 5T(-3\mathbf{e}_1 + 2\mathbf{e}_2) \\ &= 3(2\mathbf{e}_1 + 3\mathbf{e}_2) + 5(-\mathbf{e}_1 - 2\mathbf{e}_2) \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned}$$

so the standard matrix of T is

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$