UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATH 122 200610 Midterm Test 1 (S Version)

Time: 50 minutes
Instructor: Dr. Edward Doolittle

Name: ______________________
Student #: ______________________

(marks) Please do all questions. You have 50 minutes to do the exam, which is worth 50 marks; you should try to earn one mark per minute. A non-programmable calculator is allowed but is not necessary. You may leave early if you can do so without disturbing any of your colleagues. If you finish early, I suggest you check your work thoroughly.

1. (a) Solve the linear system

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 &= -2 \\ 3x_2 - 2x_3 &= 2 \\ x_1 + 5x_2 - 3x_3 &= 3 \end{aligned}$$

   Ans. $x_1 = -1/3 - (1/3)s$, $x_2 = (2/3) + (2/3)s$, $x_3 = s$.

   (b) Describe the solution set geometrically.
   Ans. Line through $(-1/3, 2/3, 0)$ parallel to the vector $(-1/3, 2/3, 1)$.

   (c) Show that the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}.$$

   Ans. One example is $x_1 = -1/3$ $x_2 = 2/3$, $x_3 = 0$.

   (d) Does the set of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ span $\mathbb{R}^3$? Justify your answer.
   Ans. No.

   (e) Is the set of vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ linearly independent? Justify your answer.
   Ans. No.

2. Consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$T(x_1, x_2) = (2x_1 - x_2, x_2, 0, -x_1 + 3x_2).$$

   (a) Show that T is a linear transformation.
   Ans. You can find this tedious calculation in the solutions to one of the problem sets. If you are confused, try showing that the mapping $T_1 : \mathbb{R}^2 \to \mathbb{R}^1$ given by $T_1(x_1, x_2) = 2x_1 - x_2$ is linear, and the same for $T_2(x_1, x_2) = x_2$, $T_3(x_1, x_2) = 0$, and $T_4(x_1, x_2) = -x_1 + 3x_2$. The calculation that you have to do is a combination of those.

   (b) Find the standard matrix of T.
   Ans.

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 0 & 0 \\ -1 & 3 \end{bmatrix}.$$

3. Consider the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 2 & -6 & 15 & 5 \\ -1 & 8 & -7 & 1 \end{bmatrix}$$

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(a) What is the domain of T? The codomain of T?
Ans. $\mathbb{R}^4$; $\mathbb{R}^3$

(b) Is T onto? Is T one-to-one? Justify your answers.
Ans. Yes; no.

4. Consider the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -1 \\ 2 & -3 & k \end{bmatrix}$$

where k is a parameter.

(a) For what values of k is T one-to-one? Justify your answer.
Ans. All values of k except $k = 5$.

(b) For what values of k is T onto? Justify your answer.
Ans. All values of k except $k = 5$.

5. Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors in $\mathbb{R}^n$, and consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. Show that if $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent, then T is not one-to-one.
Ans. The idea is $0 = x_1T(\mathbf{u}) + x_2T(\mathbf{v})$ for some non-trivial (non-zero) x_1 and x_2 by linear dependence, and $x_1T(\mathbf{u}) + x_2T(\mathbf{v}) = T(x_1\mathbf{u} + x_2\mathbf{v})$ by linearity. So we have two vectors that map to $\mathbf{0}$: $\mathbf{0} \in \mathbb{R}^n$ and $x_1\mathbf{u} + x_2\mathbf{v}$; the latter is non-zero because of linear independence.