MATH122 200610 Midterm Test 2B Solutions

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1. (a)

$$AD = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix},$$

$$DA = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}.$$

(b) Comparing A and DA, we see that the latter could be obtained from the former by multiplying the first row by 2, the second row by 3, and the third row by 5. Therefore D should be the product of the elementary matrices which correspond to those row operations, i.e., D should be equal to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

You should check the result by matrix multiplication.

2. (a) Find the inverse by performing row reduction on the matrix

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right].$$

Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right].$$

Adding -4 times row 1 to row 3 gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right].$$

Adding 3 times row 2 to row 3 gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right].$$

Multiplying row 1 by 2 gives

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right].$$

Adding -1 times row 3 to row 2 and -3 times row 3 to row 1 gives

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -9 & 14 & -3 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right].$$

Multiplying rows 1 and 3 by 1/2 gives the inverse

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

You should check the inverse by matrix multiplication (on either side) with A. (Why don't you have to check by multiplication on both sides?)

(b) The equation cannot have no solutions, and cannot have more than one solution any $\mathbf{b}$ by the invertible matrix theorem (Theorem 8). The transformation T is onto and one-to-one again by the invertible matrix theorem.

(c) Once we have the inverse we can solve the equation $A\mathbf{x} = \mathbf{b}$ easily and quickly by performing the matrix multiplication $\mathbf{x} = A^{-1}\mathbf{b}$:

$$\mathbf{x} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 27 \\ 16 \\ -7 \end{bmatrix}.$$

Check the result by multiplying $A\mathbf{x}$:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}\begin{bmatrix} 27 \\ 16 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix},$$

as required.

3. (a) Matrix operations follow the same order of operations conventions as ordinary arithmetic operations, so the given expression should be bracketed as

$$((B + (4I))B) - (21I).$$

We have

$$B - 4I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix},$$

so

$$(B + 4I)B = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix}$$

and

$$(B + 4I)B - 21I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) By the above we have

$$(B + 4I)B - 21I = O$$

where O is the matrix of all zeros. Rearranging,

$$\frac{1}{21}(B + 4I)B = I$$

so

$$B^{-1} = \frac{1}{21}(B + 4I).$$

There's no need to write out the entries of B^{-1} in detail, but if you wish, the result is

$$B^{-1} = \begin{bmatrix} 6/21 & 3/21 \\ 3/21 & -2/21 \end{bmatrix}$$

or, in lowest terms,

$$B^{-1} = \begin{bmatrix} 2/7 & 1/7 \\ 1/7 & -2/21 \end{bmatrix}.$$

Alternatively, you can use Cramer's rule to calculate the entries of B^{-1}.

4. (a) To find a basis for the null space, we solve the system $A\mathbf{x} = \mathbf{0}$. We do so by row reducing the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ to *reduced* row echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & -9 & 0 \\ -2 & -2 & 2 & -8 & 2 & 0 \\ 2 & 3 & 0 & 7 & 1 & 0 \\ 3 & 4 & -1 & 11 & -8 & 0 \end{array}\right].$$

Adding 2 times row 1 to row 2, -2 times row 1 to row 3, and -3 times row 1 to row 4 gives

$$\left[\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & -9 & 0 \\ 0 & 4 & 8 & -4 & -16 & 0 \\ 0 & -3 & -6 & 3 & 19 & 0 \\ 0 & -5 & -10 & 5 & 19 & 0 \end{array}\right].$$

Multiplying row 2 by 1/4 gives

$$\left[\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & -9 & 0 \\ 0 & 1 & 2 & -1 & -4 & 0 \\ 0 & -3 & -6 & 3 & 19 & 0 \\ 0 & -5 & -10 & 5 & 19 & 0 \end{array}\right].$$

Adding 3 times row 2 to row 3 and 5 times row 2 to row 4 gives

$$\left[\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & -9 & 0 \\ 0 & 1 & 2 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

Multiplying row 3 by 1/7 and then adding the appropriate multiples of row 3 to the other rows gives

$$\left[\begin{array}{ccccc|c} 1 & 3 & 3 & 2 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -3 times row 2 to row 1 gives

$$\left[\begin{array}{ccccc|c} 1 & 0 & -3 & 5 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Now that the system is in reduced row echelon form we can read off the solution directly:

$$\begin{aligned} x_1 &= 3s - 5t \\ x_2 &= -2s + t \\ x_3 &= s \\ x_4 &= t \\ x_5 &= 0, \end{aligned}$$

or, in vector notation,

$$\mathbf{x} = s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

In summary, a basis for the null space of A is given by the vectors

$$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

You should check that those vectors really are in the null space by matrix multiplication.

(b) A basis for the column space of A is given by its pivot columns. Label the columns $\mathbf{v}_1, \ldots, \mathbf{v}_5$. By the above row reduction, the pivot columns of A are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_5$. To find the linear combinations, we use the null space basis computed in the previous problem. The null space basis vectors tell us that

$$\begin{aligned} 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 &= \mathbf{0} \\ -5\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4 &= \mathbf{0} \end{aligned}$$

so

$$\begin{aligned} \mathbf{v}_3 &= -3\mathbf{v}_1 + 2\mathbf{v}_2 \\ \mathbf{v}_4 &= 5\mathbf{v}_1 - \mathbf{v}_2. \end{aligned}$$

You should check the above results using vector arithmetic. That check, plus the check in part (a), provides a high level of confidence that the answer is correct.

5. We solve this matrix equation in much the same way as we would solve the corresponding equation in ordinary algebra, except that we have to be careful about the order in which we multiply matrices. (Matrix multiplication is not commutative, so $MN \neq NM$ in general.) The cleanest and quickest way to solve this problem is to take the inverse of both sides:

$$((A - AX)^{-1})^{-1} = (X^{-1}B)^{-1}.$$

Using the rules $(M^{-1})^{-1} = M$ and $(MN)^{-1} = N^{-1}M^{-1}$ for any matrices M and N (note the change in order of multiplication), we have

$$A - AX = B^{-1}X.$$

(For a complete answer, you should justify why B is invertible. One way to do so is to note that $B = X(A - AX)^{-1}$ so is a product of two invertible matrices, so is invertible.) Now we solve for X:

$$\begin{aligned} A &= AX + B^{-1}X \\ A &= (A + B^{-1})X \\ X &= (A + B^{-1})^{-1}A. \end{aligned}$$

(Again, for a complete answer, we need to justify why $A + B^{-1}$ is invertible. If it were not invertible, then $A = (A + B^{-1})X$ would not be invertible, contradicting the information which was supplied.)

An important point in the above calculation is that we get an equation of the form $M_1X + M_2X = M_3$ which can be factored to give $(M_1 + M_2)X = M_3$. If we had an equation of the form $M_1X + XM_2 = M_3$, we could not factor, and in fact the latter equation is considerably more difficult to solve in general.