MATH122 200610 Problem Set 1 Solutions DRAFT

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1. (a) The augmented matrix corresponding to the system is

$$\left[\begin{array}{cc|c} 2 & 4 & -4 \\ 5 & 7 & 11 \end{array}\right].$$

Adding −2 times row 1 to row 2 we obtain

$$\left[\begin{array}{cc|c} 2 & 4 & -4 \\ 1 & -1 & 19 \end{array}\right].$$

Swapping row 1 and row 2 we obtain

$$\left[\begin{array}{cc|c} 1 & -1 & 19 \\ 2 & 4 & -4 \end{array}\right].$$

Adding −2 times row 1 to row 2 we obtain

$$\left[\begin{array}{cc|c} 1 & -1 & 19 \\ 0 & 6 & -42 \end{array}\right].$$

Dividing row 2 through by 6 we obtain

$$\left[\begin{array}{cc|c} 1 & -1 & 19 \\ 0 & 1 & -7 \end{array}\right].$$

Dividing row 2 through by 6 we obtain

$$\left[\begin{array}{cc|c} 1 & -1 & 19 \\ 0 & 1 & -7 \end{array}\right].$$

Adding row 2 to row 1 we finally obtain

$$\left[\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & -7 \end{array}\right].$$

The solution is $x_1 = 12$, $x_2 = -7$.

(b) The augmented matrix corresponding to the system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right].$$

Adding −3 times row 1 to row 2 and 4 times row 1 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right].$$

Adding 3 times row 2 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right].$$

The system is now in row echelon form. We could continue from this point to put the system into reduced row echelon form, but that is not necessary. We can see from the row echelon form that the system is inconsistent (has no solutions) because the final row of the augmented matrix corresponds to the equation $0 = 3$ which cannot be solved. Therefore the original system also has no solutions.

2. The matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

is already in row echelon form, but it will be helpful if we perform further operations to put it into reduced row echelon form. We use each of the leading 1's to eliminate all elements above it in the column. Starting with the last leading 1, we add 4 times row 4 to row 2 and we add −3 times row 4 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right].$$

Now we add 2 times row 2 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right].$$

We can read the solution to the corresponding system directly from this augmented matrix: $x_1 = -3$, $x_2 = -5$, $x_3 = 6$, and $x_4 = -3$. The solution set is $\{(x_1, x_2, x_3, x_4)\} = \{(-3, -5, 6, -3)\}$. (You should check that this really is a solution to the system corresponding to the original augmented matrix.)

3. In order to solve this system we put the matrix into row echelon form. We treat h as if it were a number. Starting with the original system

$$\left[\begin{array}{cc|c} 1 & h & -3 \\ -2 & 4 & 6 \end{array}\right]$$

we add 2 times row 1 to row 2 to obtain

$$\left[\begin{array}{cc|c} 1 & h & -3 \\ 0 & 4+2h & 0 \end{array}\right].$$

The final row of the above augmented matrix corresponds to the equation $(4 + 2h)x_2 = 0$. That equation always has a solution (namely $x_2 = 0$) so the original system is always consistent, with the solution $x_1 = -3$, $x_2 = 0$.

4. The obvious solution to this problem is the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We can obtain other augmented matrices which are row equivalent to the given matrix (and hence correspond to systems with the same solution set as the given system) by performing any random row operation to the above matrix. For example, if we add 3 times row 1 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & -6 \end{array}\right]$$

and if we switch row 2 and row 3 in the above we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 3 & 0 & 1 & -6 \\ 0 & 1 & 0 & 1 \end{array}\right].$$

We now have three augmented matrices which correspond to systems the solution of which is $x_1 = -2$, $x_2 = 1$, and $x_3 = 0$. It should be clear how to generate many more examples.

5. The augmented matrix corresponding to the system is

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right].$$

We would like to put this system into row echelon form. However, we must be careful not to multiply or divide by 0. It is also helpful if we arrange our calculations so as to avoid fractions wherever possible, although if you don't mind working with fractions it is certainly possible to multiply row 1 by $1/a$, rather than multiplying row 2 by a, as a first step.

Since we know that a is not 0 we can multiply the second row by a to obtain

$$\left[\begin{array}{cc|c} a & b & f \\ ac & ad & ag \end{array}\right].$$

Now adding $-c$ times row 1 to row 2 we obtain

$$\left[\begin{array}{cc|c} a & b & f \\ 0 & ad-bc & ag-cf \end{array}\right].$$

The system is now in row echelon form (why?). The second row corresponds to the equation

$$(ad - bc)x_2 = ag - cf.$$

If $ad - bc \neq 0$ then that equation has a solution (namely $x_2 = (ag - cf)/(ad - bc)$) and the system is consistent. If, on the other hand, $ad - bc = 0$, then the equation is of the form $0x_2 = ag - cf$ which does not have a solution in general because $ag - cf$ is not 0 for some value of g. Therefore if $ad - bc = 0$, the system is inconsistent for some values of g.

In summary, if the system is consistent for all values of f and g, we can say that $ad - bc \neq 0$.