MATH122 200610 Problem Set 2 Solutions DRAFT

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1. The original matrix is

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}.$$

Adding -3 times row 1 to row 2 and -5 times row 1 to row 3 gives

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}.$$

Adding 2 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -58 \end{bmatrix}.$$

The matrix is now in row echelon form. (There are other possible answers at this stage of the problem.) To get it into reduced row echelon form, multiply row 2 by $-1/4$ and row 3 by $-1/58$ to obtain

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now add -3 times row 3 to row 2 and -7 times row 3 to row 1 to eliminate the elements above the leading 1 in the third row:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, add -3 times row 2 to row 1:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix is now in reduced row echelon form. There is a unique answer to this part of the question, so if you got any other result here, one of us is certainly wrong.

The pivot positions in the final matrix are the positions of the leading 1s in the reduced row echelon form. To find the pivot positions in the original matrix, follow the leading 1s backwards through the process; since no rows were swapped, the pivot positions in the original matrix are the same as the pivot positions in the final matrix (namely row 1, column 1; row 2, column 2; and row 3, column 4). The pivot columns are the columns containing pivot positions, namely columns 1, 2, and 4.

2. The original augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right].$$

Adding 1 times row 1 to row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right].$$

Adding 4 times row 2 to row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix is now in reduced row echelon form. There is no row with a leading element in the final position (i.e., the final column is not a pivot column) so the system is consistent. The two non-trivial equations in the system are

$$\begin{array}{rcrcrcrcr} x_1 & - & 7x_2 & & & + & 6x_4 & = & 5 \\ & & & & x_3 & - & 2x_4 & = & -3 \end{array}$$

The variables in non-pivot columns are free and the variables in pivot columns are determined. We set $x_2 = s$ and $x_4 = t$. Then the general solution to the system is

$$\begin{aligned}x_1 &= 5 + 7s - 6t\\ x_2 &= s\\ x_3 &= -3 + 2t\\ x_4 &= t.\end{aligned}$$

We should check that we actually do have a solution to the system. The first equation is

$$\begin{aligned}5 &= x_1 - 7x_2 + 6x_3\\ &= (5 + 7s - 6t) - 7s + 6t\\ &= 5\end{aligned}$$

which is true. The second equation is

$$\begin{aligned}-3 &= x_3 - 2x_4\\ &= (-3 + 2t) - 2t\\ &= -3\end{aligned}$$

which is also true. Finally, the third equation is

$$\begin{aligned}7 &= -x_1 + 7x_2 - 4x_3 + 2x_4\\ &= -(5 + 7s - 6t) + 7s - 4(-3 + 2t) + 2t\\ &= -5 - 7s + 6t + 7s + 12 - 8t + 2t\\ &= 7\end{aligned}$$

which is also true. This check reduces the likelihood that we have made an error in the calculations.

3. The system corresponds to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2\\ 3 & h & k \end{array}\right].$$

We reduce it to row echelon form by adding -3 times the first row to the second to obtain

$$\left[\begin{array}{cc|c} 1 & 3 & 2\\ 0 & h-9 & k-6 \end{array}\right].$$

(a) The system has a unique solution if each of the columns in the coefficient matrix are pivot columns. The first column is always a pivot column, and for example, picking $h = 10$ makes the second column a pivot column. Picking $k = 7$ gives us a nice system with unique solution $x_2 = 1$, $x_1 = 2 - 3(1) = -1$. (Any other answers can be obtained by picking a value of h different from 9 and any value of k at all.)

(b) The system has no solution if there is a row of the form $\left[\begin{array}{cc|c} 0 & 0 & \square \end{array}\right]$ where $\square$ is a non-zero number. That is only possible in our example if $h-9 = 0$ and $k-6 \neq 0$. So for example, picking $h = 9$ and $k = 7$ would give us a system with no solution. (Any other answer is of can be obtained by picking $h = 9$ and any value of k different from 6.)

(c) The system has infinitely many solutions if there is a free variable, i.e., a column in the coefficient matrix is not a pivot column, and if the system is consistent. The only way to arrange such a situation is by taking $h = 9$ and $k = 6$.

4. The vector $\mathbf{b}$ is a linear combination of the other three vectors if and only if there is a solution to the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5\\ -2 & 5 & 0 & 11\\ 2 & 5 & 8 & -7 \end{array}\right].$$

We now row reduce the above matrix. Add 2 times row 1 to row 2 and -2 times row 1 to row 3 to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5\\ 0 & 5 & 4 & 1\\ 0 & 5 & 4 & 3 \end{array}\right].$$

Now add -1 times row 2 to row 3 to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5\\ 0 & 5 & 4 & 1\\ 0 & 0 & 0 & 2 \end{array}\right].$$

The matrix is now in row echelon form. Since the final column of the augmented matrix is a pivot column, the corresponding system has no solution, and we conclude that $\mathbf{b}$ cannot be expressed as a linear combination of the other three vectors.

5. The vector $\mathbf{y}$ is in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the system corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & h\\ 0 & 1 & -5\\ -2 & 8 & -3 \end{array}\right]$$

is consistent. Row reduce the above matrix by adding 2 times row 1 to row 3 to obtain

$$\left[\begin{array}{cc|c} 1 & -3 & h\\ 0 & 1 & -5\\ 0 & 2 & -3+2h \end{array}\right]$$

and then by adding -2 times row 2 to row 3 to obtain

$$\left[\begin{array}{rr|r} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{array}\right]$$

which is in row echelon form. The corresponding system is consistent if and only if $7+2h=0$, i.e., if and only if $h=-7/2$, which is therefore the only value of h for which $\mathbf{y}$ is in the plane spanned by the other two vectors. (For that value of h, you should check your answer by finding x_1 and x_2 such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{y}$.)