MATH122 200610 Problem Set 3 Solutions DRAFT

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1. We look for solutions to the equation $A\mathbf{x} = \mathbf{b}$, or equivalently, solutions to the linear system corresponding to the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. To solve the system we use row reduction.

   (a) The augmented matrix in which we are interested is
   $$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right].$$
   Adding 3 times row 1 to row 2,
   $$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right].$$
   Adding -1 times row 2 to row 3,
   $$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right].$$
   The system is now in row echelon form. There is no row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \square \end{array}\right]$ where $\square$ is a non-zero number, so the system is consistent. That means that $\mathbf{b}$ can be expressed as a linear combination of the columns of A so $\mathbf{b}$ is in the subset of $\mathbb{R}^3$ spanned by the columns of A.

   It would be a good exercise for you to figure out exactly what linear combination of the columns of A is equal to $\mathbf{b}$, but it is not necessary to answer this problem.

   (b) Please check this space later.

2. The equation $A\mathbf{x} = \mathbf{b}$ does not have a solution if and only if the system corresponding to the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ does not have a solution. So we form the augmented matrix, perform row reduction, and check for a row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & \square \end{array}\right]$ in the resulting augmented matrix.

   (a) The augmented matrix is
   $$\left[\begin{array}{rrr|c} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{array}\right].$$
   Adding 3 times row 1 to row 2, and -5 times row 1 to row 3,
   $$\left[\begin{array}{rrr|r} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{array}\right].$$
   Adding 2 times row 2 to row 3,
   $$\left[\begin{array}{rrr|r} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & (b_3 - 5b_1) + 2(b_2 + 3b_1) \end{array}\right].$$
   A little algebra gives
   $$\left[\begin{array}{rrr|r} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{array}\right]$$
   as the system. We can arrange it so that this system does not have a solution if, for example, $b_1 = 1$, $b_2 = 0$, $b_3 = 0$ in which case the last row of the augmented matrix corresponds to an inconsistent equation. The equation $A\mathbf{x} = \mathbf{b}$ does have a solution if the final row corresponds to a consistent equation, i.e., for all $\mathbf{b}$ with $b_1 + 2b_2 + b_3 = 0$.

   (b) The augmented matrix is
   $$\left[\begin{array}{rrr|c} 0 & 5 & -2 & b_1 \\ 4 & 8 & 3 & b_2 \\ -4 & 2 & -7 & b_3 \end{array}\right].$$
   We would like to get the pivot position in the first column at the first row, but

since there is a 0 there we must swap two rows. Swapping row 1 and row 2,

$$\left[\begin{array}{rrr|l} 4 & 8 & 3 & b_2 \\ 0 & 5 & -2 & b_1 \\ -4 & 2 & -7 & b_3 \end{array}\right].$$

Now adding 1 times row 1 to row 3,

$$\left[\begin{array}{rrr|r} 4 & 8 & 3 & b_2 \\ 0 & 5 & -2 & b_1 \\ 0 & 10 & -4 & b_3 + b_2 \end{array}\right].$$

Adding -2 times row 2 to row 3,

$$\left[\begin{array}{rrr|r} 4 & 8 & 3 & b_2 \\ 0 & 5 & -2 & b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 + b_3 \end{array}\right].$$

The corresponding system is inconsistent if, for example, $b_1 = 0$, $b_2 = 0$, and $b_3 = 1$, so for the vector $\mathbf{b}$ with those components, the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution. The condition for consistency is $-2b_1 + b_2 + b_3 = 0$ which is a description of the set of all $\mathbf{b}$ for which the equation does have a solution.

3. Recall that a vector $\mathbf{b} \in \mathbb{R}^4$ is in the span of the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ if $\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k$ for some scalars $x_1, x_2, \ldots, x_k$. Equivalently, $\mathbf{b}$ is in the span of S if there is a solution to the equation $A\mathbf{x} = \mathbf{b}$ where A is the matrix with the $\mathbf{v}_i$ as columns. Determining whether a given vector $\mathbf{b}$ is in the span of a set of vectors S reduces to a question similar to those in question 1 of this problem set.

   We say that the set of vectors S spans $\mathbb{R}^4$ if every vector $\mathbf{b} \in \mathbb{R}^4$ is in the span of S. Equivalently, S spans $\mathbb{R}^4$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ can be solved for *every* $\mathbf{b} \in \mathbb{R}^4$ where A is again the matrix with the members of S as columns. Determining whether a set S of vectors spans $\mathbb{R}^4$ reduces to a question similar to those in question 2 of this problem set.

   (a) In this case the augmented matrix is

$$\left[\begin{array}{rrrr|l} 5 & -7 & -4 & 9 & b_1 \\ 6 & -8 & -7 & 5 & b_2 \\ 4 & -4 & -9 & -9 & b_3 \\ -9 & 11 & 16 & 7 & b_4 \end{array}\right].$$

Adding -1 times row 3 to row 1 gives

$$\left[\begin{array}{rrrr|l} 1 & -3 & 5 & 18 & b_1 - b_3 \\ 6 & -8 & -7 & 5 & b_2 \\ 4 & -4 & -9 & -9 & b_3 \\ -9 & 11 & 16 & 7 & b_4 \end{array}\right].$$

Adding -6 times row 1 to row 2, -4 times row 1 to row 3, and 9 times row 1 to row 4 gives

$$\left[\begin{array}{rrrr|l} 1 & -3 & 5 & 18 & b_1 - b_3 \\ 0 & 10 & -37 & -103 & -6b_1 + b_2 + 6b_3 \\ 0 & 8 & -29 & -81 & -4b_1 + 5b_3 \\ 0 & -16 & 61 & 169 & 9b_1 - 9b_3 + b_4 \end{array}\right].$$

Adding -1 times row 3 to row 2 gives

$$\left[\begin{array}{rrrr|l} 1 & -3 & 5 & 18 & b_1 - b_3 \\ 0 & 2 & -8 & -22 & -2b_1 + b_2 + b_3 \\ 0 & 8 & -29 & -81 & -4b_1 + 5b_3 \\ 0 & -16 & 61 & 169 & 9b_1 - 9b_3 + b_4 \end{array}\right].$$

Adding -4 times row 2 to row 3 and 8 times row 2 to row 4 gives

$$\left[\begin{array}{rrrr|l} 1 & -3 & 5 & 18 & b_1 - b_3 \\ 0 & 2 & -8 & -22 & -2b_1 + b_2 + b_3 \\ 0 & 0 & 3 & 7 & 4b_1 - 4b_2 + b_3 \\ 0 & 0 & -3 & -7 & -7b_1 + 8b_2 - b_3 + b_4 \end{array}\right].$$

Adding 1 times row 3 to row 4 gives

$$\left[\begin{array}{rrrr|l} 1 & -3 & 5 & 18 & b_1 - b_3 \\ 0 & 2 & -8 & -22 & -2b_1 + b_2 + b_3 \\ 0 & 0 & 3 & 7 & 4b_1 - 4b_2 + b_3 \\ 0 & 0 & 0 & 0 & -3b_1 + 4b_2 + b_4 \end{array}\right].$$

As in problem 2, we can find a vector $\mathbf{b}$ for which the system has no solution (e.g., setting $b_1 = 1$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0$ makes the last row of the augmented matrix inconsistent). Therefore the columns of A do not span $\mathbb{R}^4$.

   (b) We can solve this problem in a similar manner as the previous. However, there is a useful shortcut we can take. We really only need to keep track of the coefficient matrix, and not all the b's. The reason is that if there is no row of 0 in the row-reduced coefficient matrix, we can always solve the system no matter what $\mathbf{b}$ is, whereas if there is a row of 0 in the row-reduced coefficient matrix, we can construct a $\mathbf{b}$ for which there is no solution. (Doing so is a little tricky: augment the row-reduced coefficient matrix with a column with all 0's except for a

1 in the bottom entry, and then run the row reduction in reverse to construct $\mathbf{b}$.) The coefficient matrix is

$$\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}.$$

Adding 1 times row 2 to row 1 gives

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix}.$$

Adding 7 times row 1 to row 2, −11 times row 1 to row 3, and 3 times row 1 to row 4 gives

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & 13 & -2 & -1 & 19 \\ 0 & -26 & 4 & 2 & -50 \\ 0 & 13 & -2 & 5 & 19 \end{bmatrix}.$$

Adding 2 times row 2 to row 3 and −1 times row 2 to row 4,

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & 13 & -2 & -1 & 19 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix}.$$

Swapping row 3 and row 4 gives

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & 13 & -2 & -1 & 19 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix}$$

which is in row echelon form. Since there is no row of all 0 in the row-reduced coefficient matrix, we can always solve the system $A\mathbf{x} = \mathbf{b}$, so every $\mathbf{b}$ is in the span of the columns of A, i.e., the columns of A span $\mathbb{R}^4$.

4. (a) Solving $A\mathbf{x} = \mathbf{b}$ is equivalent to solving the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right].$$

Adding 4 times row 1 to row 2,

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right].$$

Adding 1 times row 2 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The matrix is now in row echelon form. Multiplying row 2 by 1/3,

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding −3 times row 2 to row 1,

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in reduced row echelon form. The variable x_3 is free, and the solution is

$$\begin{aligned} x_1 &= 5s \\ x_2 &= -2s \\ x_3 &= s, \end{aligned}$$

which in parametric form may be written

$$\mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} s.$$

(b) The solutions to $A\mathbf{x} = \mathbf{b}$ are the solutions to the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right].$$

Adding −1 times row 1 to row 2 and 3 times row 1 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right].$$

Adding −2 times row 2 to row 3,

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -3 times row 2 to row 1,

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution to the corresponding system is

$$\begin{aligned} x_1 &= -5 - 4s \\ x_2 &= -3 + 3s \\ x_3 &= s \end{aligned}$$

which may be written

$$\mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} s.$$

5. The system is already in reduced row echelon form. The variables x_2 and x_3 are free so we can write the general solution as

$$\begin{aligned} x_1 &= 3s - 5t \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

which may be written in parametric form as

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} t.$$

Similarly, the solution set to the second equation may be written in parametric form as

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} t.$$

The solution set to the first equation is a plane through the origin and the points $(3, 1, 0)$ and $(-5, 0, 1)$. The solution set to the second equation is a plane parallel to the the plane of the solution set to the first equation, but passing through the point $(4, 0, 0)$ instead of the origin.