MATH122 200610 Problem Set 4 Solutions DRAFT

Edward Doolittle

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1. (a) Write the equation as $(x_1)Na_3PO_4 + (x_2)Ba(NO_3)_2 \rightarrow (x_3)Ba_3(PO_4)_2 + (x_4)NaNO_3$. Note that $(NO_3)_2$ means NO_3NO_3, i.e., two N's and six O's. Conservation of Na gives the equation $3x_1 = x_4$. Conservation of P gives $x_1 = 2x_3$. Conservation of O gives $4x_1+6x_2 = 8x_3+3x_4$. Conservation of Ba gives $x_2 = 3x_3$. Conservation of N gives $2x_2 = x_4$. In augmented matrix form, our system of equations becomes

$$\left[\begin{array}{rrrr|r} 3 & 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right].$$

Swapping row 1 and row 2 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right].$$

Adding -3 times row 1 to row 2 and -4 times row 1 to row 3 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 6 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right].$$

Swapping row 2 and row 4 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 6 & 0 & -3 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right].$$

Adding -6 times row 2 to row 3 and -2 times row 2 to row 5 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 18 & -3 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right].$$

Multiplying row 3 by 1/3 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right].$$

Adding -1 times row 3 to row 4 and -1 times row 3 to row 5 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is in row echelon form. Multiplying row 3 by 1/6 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding 3 times row 3 to row 2 and 2 times row 3 to row 1 gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -2/6 & 0 \\ 0 & 1 & 0 & -3/6 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution to the system is

$$x_1 = \frac{1}{3}s$$
$$x_2 = \frac{1}{2}s$$
$$x_3 = \frac{1}{6}s$$
$$x_4 = s$$

The smallest positive integer value of s that makes all the x_i integers is $s = 6$, in which case we have $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, and $x_4 = 6$. You should check that those numbers balance the equation.

(b) The augmented matrix associated with this problem is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -1 & 0 \\ 0 & 35 & 4 & -4 & 0 & -12 & -1 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Adding -1 times row 1 to row 2 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -1 & 0 \\ 0 & 35 & 4 & -4 & 0 & -12 & -1 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Adding -17 times row 3 to row 5 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Swapping row 2 and row 5 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Adding -2 times row 2 to row 3 and -10 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & -8 & 8 & -35 & 24 & 2 \\ 0 & 0 & -40 & 40 & -170 & 119 & 10 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Swapping row 3 and row 5 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & -40 & 40 & -170 & 119 & 10 \\ 0 & 0 & -8 & 8 & -35 & 24 & 2 \\ 0 & 0 & 2 & 0 & -3 & 0 & -2 \end{bmatrix}.$$

Adding 40 times row 3 to row 4, 8 times row 3 to row 5, and -2 times row 3 to row 6 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 80 & -170 & -1 & 10 \\ 0 & 0 & 0 & 16 & -35 & 0 & 2 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \end{bmatrix}.$$

Swapping row 4 and row 6 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 16 & -35 & 0 & 2 \\ 0 & 0 & 0 & 80 & -170 & -1 & 10 \end{bmatrix}.$$

Adding 8 times row 4 to row 5, and 40 times row 4 to row 6, gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & -59 & 48 & -14 \\ 0 & 0 & 0 & 0 & -290 & 239 & -70 \end{bmatrix}.$$

Adding -5 times row 5 to row 6,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & -59 & 48 & -14 \\ 0 & 0 & 0 & 0 & 5 & -1 & 0 \end{bmatrix}.$$

Adding 11 times row 6 to row 5,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & -4 & 37 & -14 \\ 0 & 0 & 0 & 0 & 5 & -1 & 0 \end{bmatrix}.$$

Adding 1 times row 5 to row 6,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & -4 & 37 & -14 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \end{bmatrix}.$$

Swapping row 5 and row 6,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & -4 & 37 & -14 \end{bmatrix}.$$

Adding 4 times row 5 to row 6,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -3 & 6 & -2 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

The system is now in row echelon form. In the hope of working with integers for as long as possible, we can add 1 times row 5 to row 4 to obtain

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & -2 & 42 & -16 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Now multiplying row 4 by $-1/2$ gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & -4 & 17 & -12 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & -21 & 8 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Usually at this point we would multiply row 6 by $1/181$, but I am trying to avoid fractions as long as possible. So let's work from the left to the right eliminating nonzero elements above the pivot positions. Adding -4 times row 3 to row 2 gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -8 & 17 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & -21 & 8 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Adding 1 times row 4 to row 1, 8 times row 4 to row 2, and -1 times row 4 to row 3 gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -21 & 8 \\ 0 & 1 & 0 & 0 & 25 & -168 & 63 \\ 0 & 0 & 1 & 0 & -1 & 18 & -8 \\ 0 & 0 & 0 & 1 & 1 & -21 & 8 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Now adding -1 times row 5 to row 1, -25 times row 5 to row 2, 1 times row 5 to row 3, and -1 times row 5 to row 4 gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -57 & 22 \\ 0 & 1 & 0 & 0 & 0 & -1068 & 413 \\ 0 & 0 & 1 & 0 & 0 & 54 & -22 \\ 0 & 0 & 0 & 1 & 0 & -57 & 22 \\ 0 & 0 & 0 & 0 & 1 & 36 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Now, to maintain integers, instead of dividing the last row by 181, we can multiply every other row by 181:

$$\begin{bmatrix} 181 & 0 & 0 & 0 & 0 & -10317 & 3982 \\ 0 & 181 & 0 & 0 & 0 & -193308 & 74753 \\ 0 & 0 & 181 & 0 & 0 & 9774 & -3982 \\ 0 & 0 & 0 & 181 & 0 & -10317 & 3982 \\ 0 & 0 & 0 & 0 & 181 & 6516 & -2534 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Now we can use the matrix before the last as a guide to the remaining operations. Adding 57 times row 6 to row 1, 1068 times row 6 to row 2, etc., we have

$$\begin{bmatrix} 181 & 0 & 0 & 0 & 0 & 0 & -8 \\ 0 & 181 & 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 181 & 0 & 0 & 0 & -202 \\ 0 & 0 & 0 & 181 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 181 & 0 & -14 \\ 0 & 0 & 0 & 0 & 0 & 181 & -70 \end{bmatrix}.$$

Since 181 is prime, the smallest solution to the system in which all values are positive integers is $x_1 = 8$, $x_2 = 7$, $x_3 = 202$, $x_4 = 8$, $x_5 = 14$, $x_6 = 70$, and $x_7 = 181$. Check!

As you can see this problem requires industrial-strength computation. Normally calculations like this would be done with a computer, but doing them by hand illustrates some tricks that may be useful in other contexts. I used the row operations calculator at `http://www.sci.wsu.edu/math/faculty/genz/220v/lessons/kentler/rowOps.html` to help me with my calculations, though. You shouldn't waste too much time on this problem; I just included it to illustrate the scale of some of these calculations, the tricks that can be used to keep the arithmetic simple, and the hopelessness of trying to guess the answer.

2. The columns of a matrix are linearly independent if and only if the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, i.e., whether the solution to that system is unique. The solution is unique if and only if there are no free variables in the solution, which we can detect by putting the coefficient matrix into row echelon form.

   (a) Swapping row 1 and row 3, the coefficient matrix becomes

$$\begin{bmatrix} 2 & -8 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & -3 \end{bmatrix}.$$

   The matrix is in row echelon form and we can see that every column is a pivot column so there are no free variables. The solution $\mathbf{x} = \mathbf{0}$ is therefore unique, i.e., there is no non-trivial solution to $A\mathbf{x} = \mathbf{0}$, i.e., the columns of the original matrix are linearly independent.

   (b) Swapping row 1 and row 3 gives

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix}.$$

   Adding 4 times row 1 to row 3 and -5 times row 1 to row 4 gives

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix}.$$

   Adding -3 times row 2 to row 3 and 4 times row 2 to row 4 gives

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & -7 \end{bmatrix}.$$

   Swapping row 3 and row 4 gives

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}.$$

   The matrix is in row echelon form and we can clearly see that every column is a pivot column, so the columns of the original matrix are linearly independent.

   (c) Adding 3 times row 1 to row 2 gives

$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & -2 & 8 & -4 \\ 0 & 1 & -4 & 3 \end{bmatrix}.$$

   Multiplying row 2 by $-1/2$ gives

$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}.$$

   Adding -1 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & -3 & 3 & -2 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   The matrix is in row echelon form. We can see that there is a free variable $x_3 = s$ in the solution, so the columns of the original matrix are not linearly independent. (You could have also answered this question immediately by noting that we are given four vectors in $\mathbb{R}^3$, so one of our theorems tells us that they must be linearly dependent.)

3. The only case of interest here is 2(c). We must find an explicit non-trivial solution to $A\mathbf{x} = \mathbf{0}$. So all the work we did in solution 2(c) above was not in vain, after all. In order to find an explicit solution we continue the process to

put the matrix into reduced row echelon form. Adding -2 times row 3 to row 2 and 2 times row 3 to row 1 gives

$$\begin{bmatrix} 1 & -3 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Adding 3 times row 2 to row 1 gives

$$\begin{bmatrix} 1 & 0 & -9 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

All solutions to the system $A\mathbf{x} = \mathbf{0}$ are given by

$$\begin{aligned} x_1 &= 9s \\ x_2 &= 4s \\ x_3 &= s \\ x_4 &= 0. \end{aligned}$$

If we want to find single, explicit, non-trivial solution we can take s equal to any number except 0; take $s = 1$, for example, to obtain $x_1 = 9$, $x_2 = 4$, $x_3 = 1$, and $x_4 = 0$. You should check that we really do have a non-trival relation of the form $9\mathbf{v}_1 + 4\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{0}$ where $\mathbf{v}_i$ are the columns of the original matrix.

4. Recall that a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly dependent if and only if there is a set of numbers $x_1, x_2, \ldots, x_k$, not all zero, such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$. The condition is equivalent to the existence of a non-trivial solution to the matrix equation $A\mathbf{x} = \mathbf{0}$ where $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$ is the matrix formed by juxtaposing the column vectors $\mathbf{v}_i$. We search for solutions to the matrix equation by putting the matrix into row echelon form. This problem is the same as the previous problem, except it is complicated slightly by the appearance of the unknown h.

   (a) The matrix A in this case is

   $$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix}.$$

   Adding 5 times row 1 to row 2 and 3 times row 1 to row 3 gives

   $$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}.$$

   The matrix is now in row echelon form no matter what the value of h. The equation $A\mathbf{x} = \mathbf{0}$ is always solvable, so we know the system is consistent, so the only question is whether there are free variables. In this case, the variable x_2 is free no matter what the value of h, so the given vectors are linear dependent for all h. (If you don't see why, try to figure out what non-trivial linear combination of the given vectors is $\mathbf{0}$.)

   (b) The matrix in this case is

   $$\begin{bmatrix} 2 & -6 & 8 \\ -4 & 7 & h \\ 1 & -3 & 4 \end{bmatrix}.$$

   Swapping row 1 and row 3 gives

   $$\begin{bmatrix} 1 & -3 & 4 \\ -4 & 7 & h \\ 2 & -6 & 8 \end{bmatrix}.$$

   Adding 4 times row 1 to row 2 and -2 times row 1 to row 3,

   $$\begin{bmatrix} 1 & -3 & 4 \\ 0 & -5 & 16+h \\ 0 & 0 & 0 \end{bmatrix}.$$

   The matrix is now in row echelon form, no matter what the value of h. Also, no matter what the value of h, the solution to the system $A\mathbf{x} = \mathbf{0}$ has x_3 as a free variable, so the solution set is infinite, so there is a non-trivial linear combination of the given vectors that is $\mathbf{0}$, so the given set is linearly dependent for all h.

   (c) The matrix in this case is

   $$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix}.$$

   Adding 1 times row 1 to row 2 and 3 times row 1 to row 3 gives

   $$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & -7 & 3+h \end{bmatrix}.$$

   Multiplying row 2 by 1/2 gives

   $$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & -7 & 3+h \end{bmatrix}.$$

Adding 7 times row 2 to row 3 gives

$$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 10+h \end{bmatrix}.$$

The matrix is now in row echelon form no matter what h is. Note that the solution to the system $A\mathbf{x} = \mathbf{0}$ has a free variable if and only if $10 + h = 0$. The given vectors are linearly dependent if and only if $h = -10$. (For $h = -10$ find a non-trivial linear combination of the vectors that is $\mathbf{0}$.)

5. (a) If the given vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, then we can see easily that $3\mathbf{v}_1 = 2\mathbf{v}_2$, so the vectors are linearly dependent; a non-trivial linear relation is given by $3\mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{0}$, for example. It is generally easy to see whether a set of two vectors is linearly dependent; we usually just multiply the first vector by the ratio of the first element of the first vector to the first element of the second vector, and if the result is equal to the second vector, the set is linearly dependent. When might we have to vary that procedure?

   (b) Theorem 8 of section 1.7 tells us that a set of $n > 2$ vectors in $\mathbb{R}^2$ must be linearly dependent.

   (c) Theorem 9 of section 1.7 tells us that any set of vectors containing the $\mathbf{0}$ vector must be linearly dependent.