MATH122 200610 Problem Set 5 Solutions DRAFT

Edward Doolittle

Wednesday, February 15, 2006

1. The vector $\mathbf{b}$ is in the range of T if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. We use row reduction to determine whether that is the case.

   (a) The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right].$$

   Adding -3 times row 1 to row 3 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right].$$

   Adding -4 times row 2 to row 3 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

   The matrix is now in row echelon form, from which we can determine two important facts: there is a solution to the system, and the solution is unique, which answers the second question posed in the problem. However, to answer the first question, we must find an explicit solution, for which we need to put the system into reduced row echelon form. Adding 4 times row 3 to row 2 and -2 times row 3 to row 1 gives

$$\left[\begin{array}{rrr|r} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

   Adding 3 times row 2 to row 1 gives

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

   So the system has the unique solution given by $x_1 = -5$, $x_2 = -3$, and $x_3 = 1$ (check), and the answer to the first part of the question is that the vector

$$\mathbf{x} = \left[\begin{array}{r} -5 \\ -3 \\ 1 \end{array}\right]$$

   satisfies the equation $T(\mathbf{x}) = \mathbf{b}$.

   (b) The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{array}\right].$$

   Adding -3 times row 1 to row 2 and 3 times row 1 to row 4 gives

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{array}\right].$$

   Multiplying row 2 by 1/2 gives

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{array}\right].$$

   Adding -1 times row 2 to row 3 and 1 times row 2 to row 4 gives

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   The augmented matrix is in row echelon form. We can already see that the variable x_3 is free, so the solution is not unique. To put it into reduced row echelon form, add 2 times row 2 to row 1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution in parametric form is

$$\mathbf{x} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

One particular solution may be found by setting $s = 0$, and all other solutions may be found by taking different values of s.

2. To solve the first part of the problem, we must solve the system $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$. To solve the second part of the problem, we must solve the system $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. Since the row operations are the same in both cases, we should try to solve the harder problem first, and we will be able to solve the easier problem immediately thereafter.

   (a) The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{array}\right].$$

Adding -1 times row 1 to row 2 and 2 times row 1 to row 4 gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right].$$

Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 9 & 18 & 9 & 2 \end{array}\right].$$

Adding 3 times row 2 to row 3 and -9 times row 2 to row 4 gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -18 & 11 \end{array}\right].$$

Adding 6 times row 3 to row 4 gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{array}\right].$$

The system is inconsistent, so $\mathbf{b}$ is not in the range of T. On the other hand, replacing the final column with a column of zeros throughout the calculation gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is consistent. Continuing to put the system into reduced row echelon form, we multiply row 3 by 1/3 to obtain

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -3 times row 3 to row 2, and -2 times row 2 to row 1, gives

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Adding -3 times row 2 to row 1 gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution to the system in parametric form gives the set of all vectors mapped to $\mathbf{0}$ by T:

$$\mathbf{x} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

   (b) The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} -9 & -4 & -9 & 4 & 3 \\ 5 & -8 & -7 & 6 & -1 \\ 7 & 11 & 16 & -9 & 2 \end{array}\right].$$

Adding 2 times row 2 to row 1 gives

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 5 & -8 & -7 & 6 & -1 \\ 7 & 11 & 16 & -9 & 2 \end{array}\right].$$

Adding -5 times row 1 to row 2 and -7 times row 1 to row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 92 & 108 & -74 & -6 \\ 0 & 151 & 177 & -121 & -5 \end{array}\right].$$

I bet you think I'm going to give in and divide by 92 and 151 at this point, but I'm not. (You can if you want, though.) Instead, I'm going to take as many row 2's as possible from row 3 to get the size of the coefficient down. The row ops calculator linked to the web site helps. In this case, I can only take one out. Adding -1 times row 2 to row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 92 & 108 & -74 & -6 \\ 0 & 59 & 69 & -47 & 1 \end{array}\right].$$

Now I take as many row 3's as possible from row 2. In this case, I can again only take one away:

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 33 & 39 & -27 & -7 \\ 0 & 59 & 69 & -47 & 1 \end{array}\right].$$

Doing that sort of thing a few more times gets me

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 2 & 6 & -8 & -68 \\ 0 & 1 & -9 & 17 & 189 \end{array}\right].$$

Multiplying row 2 by $1/2$ gives

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 1 & 3 & -4 & -34 \\ 0 & 1 & -9 & 17 & 189 \end{array}\right].$$

Adding -1 times row 2 to row 3 gives

$$\left[\begin{array}{cccc|c} 1 & -20 & -23 & 16 & 1 \\ 0 & 1 & 3 & -4 & -34 \\ 0 & 0 & -12 & 21 & 223 \end{array}\right].$$

We're now in row echelon form. To get into reduced row echelon form, we'll have to work with fractions at some point, but my game here is to avoid them as long as possible. Let's first take care of the entries above the pivot position in the second column by adding 20 times row 2 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 37 & -64 & -679 \\ 0 & 1 & 3 & -4 & -34 \\ 0 & 0 & -12 & 21 & 223 \end{array}\right].$$

Now instead of dividing row 3 by 12, let's multiply all the other rows by 12:

$$\left[\begin{array}{cccc|c} 12 & 0 & 444 & -768 & -8148 \\ 0 & 12 & 36 & -48 & -408 \\ 0 & 0 & -12 & 21 & 223 \end{array}\right].$$

Using the matrix before the last as a guide, add 3 times row 3 to row 2 and 37 times row 3 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 12 & 0 & 0 & 9 & 103 \\ 0 & 12 & 0 & 15 & 261 \\ 0 & 0 & -12 & 21 & 223 \end{array}\right].$$

Now, fractions are unavoidable, but we have managed to avoid adding or subtracting fractions. Multiplying each row by the appropriate factor gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 9/12 & 103/12 \\ 0 & 1 & 0 & 15/12 & 261/12 \\ 0 & 0 & 1 & -21/12 & -223/12 \end{array}\right],$$

not in lowest terms. The system has solution set

$$\mathbf{x} = \frac{1}{12}\begin{bmatrix} 103 \\ 261 \\ -223 \end{bmatrix} + \frac{s}{12}\begin{bmatrix} -9 \\ -15 \\ 21 \end{bmatrix}.$$

Since the system is consistent, $\mathbf{b}$ is in the range of T. Repeating the above calculation with a final column of zeros, we find that the set of all $\mathbf{x}$ that map to $\mathbf{0}$ under T is given by

$$\mathbf{x} = \frac{s}{12}\begin{bmatrix} -9 \\ -15 \\ 21 \end{bmatrix}.$$

In both of the previous expressions, the factor of $1/12$ can be absorbed into s by letting $s' = s/12$, but the $1/12$ in front of the particular solution for $A\mathbf{x} = \mathbf{b}$ can't be eliminated.

3. In each case, we just need to keep track of the images of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and we can figure out the standard matrix from that information alone.

   (a) The geometry of right triangles (the Pythagorean theorem, etc.) tells us that the image of $(1,0)$ under the transformation is $(1/\sqrt{2}, -1/\sqrt{2})$, and similarly the image of $(0,1)$ is $(1/\sqrt{2}, 1/\sqrt{2})$. Therefore the standard matrix of the mapping is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

   (Remember, we write points as rows but vectors as columns.) Compare with the rotation matrix given on page 84 of the textbook.

   (b) In this problem we don't have to do any geometry. The standard matrix is

$$\begin{aligned} \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 + 3\mathbf{e}_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

   where I have skipped over the explicit calculation of $\mathbf{e}_2 + 3\mathbf{e}_1$.

   (c) A little geometry shows that the first reflection has the effect of mapping any point (x_1, x_2) to $(-x_1, x_2)$, and the second reflection has the effect of mapping (x_1, x_2) to (x_2, x_1). To find the standard matrix, we just need to keep track of the images of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Tracing the image of the point $(1,0)$ we have first $(-1,0)$ then $(0,-1)$; tracing the image of $(0,1)$ we have first $(0,1)$ then $(1,0)$. Therefore the standard matrix is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   (Is the final result a rotation? Do you think there is a general result that may hold for pairs of reflections?)

4. (a) The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$. The mapping is linear because

$$\begin{aligned} &T(\mathbf{x}+\mathbf{y}) \\ &\quad = T(x_1+y_1, x_2+y_2) \\ &\quad = ((x_1+y_1) - (x_2+y_2), \\ &\qquad\quad -2(x_1+y_1) + (x_2+y_2), \\ &\qquad\qquad (x_1+y_1)) \\ &\quad = (x_1 - x_2, -2x_1 + x_2, x_1) \\ &\qquad + (y_1 - y_2, -2y_1 + y_2, y_1) \\ &\quad = T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

   and

$$\begin{aligned} T(c\mathbf{x}) &= T(cx_1, cx_2) \\ &= (cx_1 - cx_2, -2cx_1 + cx_2, cx_1) \\ &= c(x_1 - x_2, -2x_1 + x_2, x_1) \\ &= cT(\mathbf{x}). \end{aligned}$$

   We have

$$\begin{aligned} T(1,0) &= (1-0, -2+0, 1) = (1,-2,1) \\ T(0,1) &= (0-1, -2(0)+1, 0) = (-1,1,0) \end{aligned}$$

   so the standard matrix of T is

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (b) The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^4$. The mapping is linear by calculations completely analogous to those of the previous problem. We have

$$\begin{aligned} T(1,0) &= (2(0) - 3(1), 1 - 4(0), 0, 0) \\ &= (-3, 1, 0, 0) \\ T(0,1) &= (2(1) - 3(0), 0 - 4(1), 0, 1) \\ &= (2, -4, 0, 1), \end{aligned}$$

   so the standard matrix for T is

$$\begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   (c) The domain is $\mathbb{R}^4$, the codomain is $\mathbb{R}^1$. The mapping is linear by calculations analogous to those of part (a). We have

$$\begin{aligned} T(1,0,0,0) &= 2 \\ T(0,1,0,0) &= 0 \\ T(0,0,1,0) &= 3 \\ T(0,0,0,1) &= -4 \end{aligned}$$

so the standard matrix for the mapping is

$$\begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix}.$$

5. A linear transformation is 1-1 if and only if the columns of its standard matrix are linearly independent, and a linear transformation is onto if and only if the columns of its standard matrix span the codomain. Both of those properties are preserved under elementary row operations, so our strategy is to row reduce the standard matrix to row echelon form and then decide whether either or both of those properties holds for the matrix in row echelon form.

   (a) Adding 2 times row 1 to row 2, and -1 times row 1 to row 3 gives

   $$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

   Adding 1 times row 2 to row 3 gives

   $$\begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

   The columns are linearly independent because each column contains a pivot element. The columns do not span the codomain because there is a row without a pivot element. Therefore the transformation is 1–1 but not onto.

   (b) Swapping row 1 and row 2 gives

   $$\begin{bmatrix} 1 & -4 \\ -3 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Adding 3 times row 1 to row 2 gives

   $$\begin{bmatrix} 1 & -4 \\ 0 & -10 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Multiplying row 2 by $-1/10$ gives

   $$\begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Adding -1 times row 2 to row 4 gives

   $$\begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   There is a pivot element in each column, therefore the columns are linearly independent, therefore the mapping is 1–1. However, there is not a pivot element in each row, therefore the columns do not span the codomain, therefore the mapping is not onto.

   (c) The standard matrix of the transformation,

   $$\begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix},$$

   is already in row echelon form. There is not a pivot element in each column, therefore the columns are not linearly independent, therefore the mapping is not 1–1. There is a pivot element in each row, therefore the columns do span the codomain, therefore the mapping is onto.