MATH122 200610 Problem Set 6 Solutions DRAFT

Edward Doolittle

Wednesday, March 1, 2006

1. (a)

$$\begin{aligned} & A + 2B \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + 2 \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}. \end{aligned}$$

(b) The operation cannot be performed because $3C$ is a 2×2 matrix, while E is a 2×1 matrix, and it is impossible to add matrices with unequal sizes.

(c)

$$\begin{aligned} & CB \\ &= \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1(7) + 2(1) & 1(-5) + 2(-4) & 1(1) + 2(-3) \\ -2(7) + 1(1) & -2(-5) + 1(-4) & -2(1) + 1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix} \end{aligned}$$

(d) It is not possible to multiply E times B, because E is a 2×1 matrix, and B is a 2×3 matrix, but in order to multiply two matrices the number of columns of the first matrix must equal the number of rows of the second matrix, and that is not true in this case. Nor is it possible to multiply BE; why not?

2. If we can multiply BC, then we know that B is of size $m \times n$, C is of size $n \times p$, and BC is of size $m \times p$. Since we know BC has size 3×4 we know $m = 3$ and $p = 4$, so B has 3 rows and C has 4 columns. From the given information we don't know anything about n.

3. Checking,

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2(8) + (-3)(5) & 2(4) + (-3)(5) \\ -4(8) + 6(5) & -4(4) + 6(5) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}, \end{aligned}$$

and

$$
\begin{aligned}
AC &= \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2(5)+(-3)(3) & 2(-2)+(-3)(1) \\ -4(5)+6(3) & -4(-2)+6(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}.
\end{aligned}
$$

So $AB = AC$ but $B \neq C$, so A cannot be cancelled from the equation. Note that A is not invertible (e.g., $\det A = 0$). In fact, you can cancel A from the equation $AB = AC$ if A is invertible (just multiply the equation on the left by A^{-1}), but if it is not invertible you cannot cancel.

4. We need to find a non-zero solution to the matrix equation $AB = O$ where O is the 2×2 zero matrix, i.e., the matrix with all 0 entries. To do this, we concentrate on one column of B at a time. We need to solve the system

$$\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and we also need to solve the system

$$\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the systems are the same, we will just solve the first one. The system corresponds to the augmented matrix

$$\left[\begin{array}{rr|r} 3 & -6 & 0 \\ -1 & 2 & 0 \end{array}\right].$$

Swapping row 1 and row 2 gives

$$\left[\begin{array}{rr|r} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right].$$

Adding 3 times row 1 to row 2 gives

$$\left[\begin{array}{rr|r} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The matrix is now in row echelon form Multiplying row 1 by -1 puts the matrix into reduced row echelon form:

$$\left[\begin{array}{rr|r} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

from which we have the solution $b_{11} = 2s$, $b_{21} = s$. Taking s to be any value other than 0 gives a non-trivial solution; e.g., we could take $s = 1$ to get the solution $b_{11} = 2$, $b_{21} = 1$. Similarly, we could solve the system for b_{12} and b_{22} to obtain, e.g., $b_{12} = -2$ and $b_{21} = -1$ (just for the sake of variety). That gives a matrix B of the form

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

You should check that $AB = O$. Try to find other possibilities for B; it should be possible to write all possible Bs in terms of two parameters s and t: try it!

Again, the key to this problem is that A is not invertible. If A is invertible, then $AB = O$ implies that $B = 0$, as you can see by multiplying the equation on the left by A^{-1}.

5. The columns of B are linearly dependent if and only if there is a non-trivial solution to the equation $B\mathbf{x} = \mathbf{0}$. Multiplying on the left by A we have $A(B\mathbf{x}) = A\mathbf{0}$ and using associativity and the fact that any matrix multiplied by a zero matrix gives a zero matrix to obtain $(AB)\mathbf{x} = \mathbf{0}$. Since $\mathbf{x}$ is non-trivial, that shows that the columns of AB are linearly dependent (with the same linear relation as the columns of B). You should try working out some specific numerical examples; create or find a matrix B with linearly dependent columns, multiply B on the left by another matrix A, and check that the columns of AB are linearly dependent with the same linear relationship as the columns of B.

6. The product $\mathbf{u}^T\mathbf{v}$ is defined because the first matrix is of size $1 \times n$ and the second is of size $n \times 1$, so they are of compatible sizes for matrix multiplication. Similarly the product $\mathbf{v}^T\mathbf{u}$ is defined. In both cases, the size of the resulting matrix is 1×1; the two are related by being transposes of one another because $(\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$. However, the transpose of a 1×1 matrix is itself, so in fact we can say that $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T\mathbf{u}$. You should try working out a numerical example; just pick two random vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ for some $n > 1$.

   Similarly, $\mathbf{v}\mathbf{u}^T$ and $\mathbf{u}\mathbf{v}^T$ both exist and are $n \times n$ matrices. The two are transposes of one another (why?) but are not equal in general. You should again pick some random vectors and try the multiplication to see what happens in this case.