MATH122 200610 Problem Set 7 Solutions DRAFT

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Wednesday, March 8, 2006

1. (a) The system may be written $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

Normally we would solve such a system using row reduction, but we now have a new tool available: matrix inversion by Cramer's rule. Applying Cramer's rule, we have $\det A = 3 \cdot 4 - 2 \cdot 7 = -2$ and

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$$

so $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$, i.e.,

$$\mathbf{x} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \end{bmatrix}$$

You should check that that value of $\mathbf{x}$ satisfies the system.

(b) In this case

$$A = \begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$$

so $\det A = 3 \cdot -8 - (-4) \cdot 7 = 4$,

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/4 \end{bmatrix}.$$

Again, you should check that that value of $\mathbf{x}$ satisfies the system.

(c) The determinant of A is $\det A = 8 \cdot -5 - 5 \cdot -7 = -5$ so

$$\mathbf{x} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

As usual, check the answer!

2. (a) Perform row reduction on the augmented matrix

$$\left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right].$$

Adding -1 times row 2 to row 1 gives

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & -1 \\ 4 & 7 & 0 & 1 \end{array}\right].$$

Adding -4 times row 1 to row 2 gives

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & -1 \\ 0 & -5 & -4 & 5 \end{array}\right].$$

Multiplying row 1 by 5 gives

$$\left[\begin{array}{cc|cc} 5 & 15 & 5 & -5 \\ 0 & -5 & -4 & 5 \end{array}\right].$$

Adding 3 times row 2 to row 1 gives

$$\left[\begin{array}{cc|cc} 5 & 0 & -7 & 10 \\ 0 & -5 & -4 & 5 \end{array}\right].$$

Multiplying row 1 by $1/5$ and row 2 by $-1/5$ gives

$$\left[\begin{array}{cc|cc} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{array}\right].$$

Therefore the inverse of the given matrix is

$$\begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}.$$

You should check the answer by multiplying it against the given matrix; you should of course obtain the identity matrix as the result.

(b) The augmented matrix of interest is

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right].$$

Adding -4 times row 1 to row 2 and 2 times row 1 to row 3 gives

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right].$$

Adding -2 times row 2 to row 3 gives

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right].$$

Since the given matrix A cannot be row equivalent to the identity matrix I (because we see from the above that A has only two pivot columns, for example, while I has three pivot columns), we can conclude that A is not invertible.

(c) The augmented matrix of interest is

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Adding -1 times row 1 to row 2 and -1 times row 1 to row 3 gives

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right].$$

Adding -1 times row 2 to row 3 gives

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & -1 & 1 \end{array}\right].$$

Multiplying row 2 by $1/2$ and row 3 by $1/3$ gives

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 3 & 0 & -1/3 & 1/3 \end{array}\right].$$

You should check that the matrix on the right of the vertical bar is the inverse of the given matrix (multiplying the two matrices should yield the identity matrix).

3. (a) The given matrix is not invertible because its columns are not linearly independent (Theorem 8(e)). (Why can we tell immediately that the columns are not linearly dependent? If you don't know, ask me.)

(b) It's not immediately obvious whether the matrix is invertible or not, so we start performing row reduction. Adding 3 times row 1 to row 3 we obtain

$$\left[\begin{array}{rrr} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{array}\right].$$

Adding 3 times row 2 to row 3 we obtain

$$\left[\begin{array}{rrr} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right].$$

There are only two pivot positions in the above matrix, so there are only two pivot positions in the given matrix, so the given matrix is not invertible by Theorem 8(c).

(c) There are four pivot positions in the given matrix, so it is invertible by Theorem 8(c). To find the inverse, we perform row operations on the augmented matrix

$$\left[\begin{array}{rrrr|rrrr} 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 0 & 5 & 9 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiplying rows 1, 2, and 3 by 5 we obtain

$$\left[\begin{array}{rrrr|rrrr} 5 & 15 & 35 & 20 & 5 & 0 & 0 & 0 \\ 0 & 25 & 45 & 30 & 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 40 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Adding -2 times row 4 to row 1, -3 times row 4 to row 2, and -4 times row 4 to row 3 we obtain

$$\left[\begin{array}{rrrr|rrrr} 5 & 15 & 35 & 0 & 5 & 0 & 0 & -2 \\ 0 & 25 & 45 & 0 & 0 & 5 & 0 & -3 \\ 0 & 0 & 10 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiplying row 1 and row 2 by 2 we obtain

$$\left[\begin{array}{rrrr|rrrr} 10 & 30 & 70 & 0 & 10 & 0 & 0 & -4 \\ 0 & 50 & 90 & 0 & 0 & 10 & 0 & -6 \\ 0 & 0 & 10 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Adding -7 times row 3 to row 1 and -9 times row 3 to row 2 we obtain

$$\left[\begin{array}{rrrr|rrrr} 10 & 30 & 0 & 0 & 10 & 0 & -35 & 24 \\ 0 & 50 & 0 & 0 & 0 & 10 & -45 & 30 \\ 0 & 0 & 10 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Multiplying row 2 by 1/5 we obtain

$$\left[\begin{array}{cccc|cccc} 10 & 30 & 0 & 0 & 10 & 0 & -35 & 24 \\ 0 & 10 & 0 & 0 & 0 & 2 & -9 & 6 \\ 0 & 0 & 10 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

Adding -3 times row 2 to row 3 we obtain

$$\left[\begin{array}{cccc|cccc} 10 & 0 & 0 & 0 & 10 & -6 & -8 & 6 \\ 0 & 10 & 0 & 0 & 0 & 2 & -9 & 6 \\ 0 & 0 & 10 & 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 \end{array}\right].$$

(You should be able to check your answer at this point if you compare with an example I did in the lectures.) Finally, multiplying each row by 1/10 gives

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1.0 & -0.6 & -0.8 & 0.6 \\ 0 & 1 & 0 & 0 & 0 & 0.2 & -0.9 & 0.6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0.5 & -0.4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

You must check the answer to have any confidence that the result is correct.

4. (a) The standard matrix for T is

$$A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}.$$

Since $\det A = 6 \cdot 7 - (-8) \cdot (-5) = 2 \neq 0$, A is invertible, so T is also invertible. By Cramer's rule and the rule for inverting linear transformations we have

$$T^{-1}(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or, in the same form in which T is given,

$$T^{-1}(x_1, x_2) = (\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2).$$

You should check that you really do have an inverse for T.

(b) Here the standard matrix for T is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -2 & 3 \end{bmatrix}.$$

Using Theorem 8 we can see that A is invertible: e.g., because A^T is in row echelon form and has a pivot in each row, A^T is invertible by Theorem 8(c); it follows that A is invertible by Theorem 8(l). From that it follows that T is invertible. To find the inverse of T, we find the inverse of A by row reduction on the matrix

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Adding -1 times row 1 to rows 2 and 3 gives

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array}\right].$$

Adding -1 times row 2 to row 3 gives

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 & -1 & 1 \end{array}\right].$$

Mulitplying row 2 by $-1/2$ and row 3 by $1/3$ give

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 1/3 \end{array}\right].$$

Check by matrix multiplication that you actually do have an inverse for A. Then the inverse of T is given by $T(\mathbf{x}) = A^{-1}\mathbf{x}$ where

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}.$$

Find a formula for T^{-1} in the same form in which T is given.

(c) The standard matrix for T is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}.$$

There is no obvious way to decide whether A is invertible or not, so we just proceed with the row reduction. We will perform row reduction on the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ so that our calculations are more useful if A turns out to be invertible. In detail, the augmented matrix in question is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -3 & -1 & 2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right].$$

Adding 3 times row 1 to row 2 gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right].$$

Adding -1 times row 2 to row 3 gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right].$$

A has now been transformed to row echelon form, from which we can see that A is invertible. We continue, putting A into reduced row echelon form, so we can find A^{-1}. Multiplying row 1 and row 2 by 2 gives

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 2 & 2 & 0 & 0 \\ 0 & 10 & 10 & 6 & 2 & 0 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right].$$

Adding 5 times row 3 to row 2 and 1 times row 3 to row 1 gives

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 0 & -1 & -1 & 1 \\ 0 & 10 & 0 & -9 & -3 & 5 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right].$$

Multiplying row 1 by 5 gives

$$\left[\begin{array}{ccc|ccc} 10 & 20 & 0 & -5 & -5 & 5 \\ 0 & 10 & 0 & -9 & -3 & 5 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right].$$

Adding -2 times row 2 to row 1 gives

$$\left[\begin{array}{ccc|ccc} 10 & 0 & 0 & 13 & 1 & -5 \\ 0 & 10 & 0 & -9 & -3 & 5 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right].$$

Finally, multiplying rows 1 and 2 by $1/10$ and row 3 by $-1/2$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.3 & 0.1 & -0.5 \\ 0 & 1 & 0 & -0.9 & -0.3 & 0.5 \\ 0 & 0 & 1 & 1.5 & 0.5 & -0.5 \end{array}\right].$$

We conclude from the above that

$$A^{-1} = \begin{bmatrix} 1.3 & 0.1 & -0.5 \\ -0.9 & -0.3 & 0.5 \\ 1.5 & 0.5 & -0.5 \end{bmatrix}$$

(check by matrix multiplication with A) and $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$.

5. There are many ways to use Theorem 8 to solve each of these problems. Below, I have given what I feel to be the most straightforward ways. The arguments get more sophisticated the further down the list, with the final argument by contradiction. Ask me if you don't understand the following arguments.

   (a) No, by Theorem 8(h).

   (b) No. By Theorem 8(g), C is invertible. Then by Theorem 8(f) we know that $\mathbf{x} \mapsto C\mathbf{x}$ is one-to-one, so the equation $C\mathbf{x} = \mathbf{b}$ can only have one solution.

   (c) No. If $G\mathbf{x} = \mathbf{y}$ has more than one solution, then the linear transformation defined by $T(\mathbf{x}) = G\mathbf{x}$ is not one-to-one. By Theorem 8(f), G is not invertible, so you can then conclude by Theorem 8(h) that the columns of G cannot span $\mathbb{R}^n$.

   (d) Yes. If $L\mathbf{x} = \mathbf{0}$ has only the trivial solution, L is invertible by Theorem 8(d). It then follows that the columns of L span $\mathbb{R}^n$ by Theorem 8(h).

   (e) If the columns of A are linearly independent, A is invertible by Theorem 8(e). Then by Theorem 8(j) there is a matrix C such that $CA = I$. It follows that A^2 is also invertible because we have $C^2A^2 = C(CA)A = CIA = CA = I$. Finally, since A^2 is invertible, the columns of A^2 span $\mathbb{R}^n$ by Theorem 8(h).

   (f) We argue by contradiction. Assume that B is not invertible; then by Theorem 8(d), there is a non-trivial solution $\mathbf{x} = \mathbf{x}_0$ to the equation $B\mathbf{x} = \mathbf{0}$. Then $(AB)\mathbf{x}_0 = A(B\mathbf{x}_0) = A\mathbf{0} = \mathbf{0}$, so $\mathbf{x} = \mathbf{x}_0$ is also a non-trivial solution to the equation $(AB)\mathbf{x} = \mathbf{0}$. By Theorem 8(d), AB is not invertible. Since this contradicts the given information that AB is invertible, our assumption that B is not invertible must be wrong, and we can conclude that B is invertible.